Document: Analysis Workshop (2011)
Professor: Shkoller
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Author: Jeff Irion

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1 9-12-11

Remark 1.1. ***Taylor Formula***

Let $u \in C^\infty(\mathbb{R})$. Expand about $x = 0$.

$$u(x) = \sum_{j=0}^{k} \frac{x^j}{j!} \frac{d^j u}{dx^j}(0) + \frac{x^{k+1}}{k!} \int_0^1 (1-s)^k \frac{d^{k+1}u}{dx^{k+1}}(sx)\, ds$$

Remark 1.2. ***Calculus Review***

$\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$.

$$\begin{aligned}\int_\Omega Df \cdot \mathbf{G}\, dx &= \int_\Omega \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathbf{G}_i\, dx \\ &= -\int_\Omega f \operatorname{div} \mathbf{G}\, dx + \int_{\partial\Omega} f \mathbf{G} \cdot \mathbf{n}\, dS\end{aligned}$$

Note that each integrand is a scalar. In the case of problem 4, we had $f = \frac{1}{2}|u|^2$, $\mathbf{G} = x$. **This formula is very important!!!**

$$\int_\Omega Df \cdot \mathbf{G}\, dx = -\int_\Omega f \operatorname{div} \mathbf{G}\, dx + \int_{\partial\Omega} f \mathbf{G} \cdot \mathbf{n}\, dS$$

2 9-13-11

Remark 2.1. ***General Tip: MCT***

When you see positivity and/or a series, think MCT.

Theorem 2.2. ***Egorov's Theorem***

Given: $\Omega \subset \mathbb{R}^n$ bounded, $(f_n) \subset \Omega,\ f_n \to f$ a.e., $|f| < \infty$ a.e.

Then for every $\delta > 0$ there exists $E \subset \Omega$ such that

(a) $\mu(E) < \delta$

(b) $f_n \to f$ uniformly on E^c

3 9-14-11

Remark 3.1. ***Some Inequalities***

Suppose $u \in C_0^\infty(\mathbb{R}^3)$. By Gagliardo-Nirenberg,

$$\alpha \|Du\|_{L^2} \geq \beta \|u\|_{L^6}$$

Now suppose $u \in C_0^\infty(\mathbb{R}^2)$. By the second Poincaré Inequality,

$$C\sqrt{q}\|Du\|_{L^2} \geq \|u\|_{L^q}, \qquad 1 \leq q < \infty$$

4 9-15-11

Remark 4.1.

If $u_n \rightharpoonup u$ in $\mathcal{H}$ and

$$\limsup_{n\to\infty} \|u_n\| \leq \|u\|$$

Then $u_n \to u$.

Proof. Assuming real-valued functions, we have

$$\|u_n - u\|^2 = \|u_n\|^2 + \|u\|^2 - 2\langle u_n, u\rangle \to 0$$

□

Theorem 4.2. ***Mazur's Theorem***

Read it. Do chapter 8 problem 20.

5 9-16-11

Remark 5.1.

We don't need to worry about convexity on the prelim.

Remark 5.2.

Jason Hole, jhole@math.ucdavis.edu, subject BSP

Shkoller's Remarks

- Points reside in the details.
- If you forget how to compute an integral, you can comment on why it's bounded.

6 Important

119A

- Memorize the "classification of fixed points" picture (page 137)
- Hamiltonian systems
- Conservative systems (see Midterm 2 cheat sheet)
- 119A Final Cheat Sheet
  - Memorize the Δ, τ, and $\lambda_{1,2}$ formulas
  - Memorize the polar coordinates identity:

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

  - A *Liapunov function* is basically a contour map whose local minima correspond to the fixed points
  - Memorize 1-D, 2-D, and Hopf bifurcations.

Analysis

- Memorize the definitions of point, continuous, and residue spectra.
- Memorize the Taylor remainder formula (see Remark 1.1).
- Memorize the "integration by parts" formula (see Remark 1.2).
- Memorize the Weierstrass approximation theorem and Stone-Weierstrass theorem.
  - Weierstrass approximation theorem: the set of polynomials is dense in $C([a, b])$ with respect to the uniform norm. From H&N page 40.
  - Stone-Weierstrass theorem, real version: "Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X, \mathbb{R})$ if and only if it separates points." From `http://en.wikipedia.org/wiki/Stone-Weierstrass_theorem#Stone.E2.80.93Weierstrass_theorem.2C_real_version`.
- When working on $(0, 1)$, note that $\int_0^1 \frac{1}{x}\,dx = \ln x\big|_0^1 = \infty$. Since log goes to infinity slowly, this is saying that $\frac{1}{x}$ is a cutoff function. Thus, $\frac{1}{x^{0.99}}$ is integrable, while $\frac{1}{x}$ is not.
- Embeddings/Inequalities
  - Sobolev spaces into k-times differentiable
  - Sobolev spaces into L^p (which p?)
  - Sobolev spaces into other Sobolev spaces (google Terrence Tao Sobolev, exercise 24)
- Riemann-Lebesgue Lemma (page 308)
- Uniform Boundedness Principle