Analysis Workshop Solutions

(Led by Professor Shkoller)

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Contents

9-12-11

Problem 1.1

$u \in C^{\infty}(\mathbb{R})$, spt $u \subset [-M, M]$.

$$v(x) = \begin{cases} \frac{1}{x^{k+1}} \left[u(x) - \sum_{j=0}^{k} \frac{x^j}{j!} \frac{d^j u}{dx^j}(0) \right] & x \neq 0 \\ \frac{1}{(k+1)!} \frac{d^{k+1}u}{dx^{k+1}}(0) & x = 0 \end{cases}$$

Prove that v is continuous.

$$\sup_{[-M,M]} |v(x)| \leq c \sup_{[-M,M]} \left| \frac{d^{k+1}u}{dx^{k+1}}(x) \right|$$

For $x \neq 0$, v is continuous because

$$v(x) = \frac{1}{k!} \int_0^1 (1-s)^k \frac{d^{k+1}u}{dx^{k+1}}(sx)\, ds.$$

$$\begin{aligned} \lim_{x \to 0} v(x) &= \int_0^1 (1-s) \frac{d^{k+1}u}{dx^{k+1}}(0)\, ds \\ &= \frac{d^{k+1}u}{dx^{k+1}}(0) \int (1-s)^k\, ds \\ &= \frac{d^{k+1}u}{dx^{k+1}}(0)(-(1-s)\big|_0^1) \end{aligned}$$

where we passed the limit through the integral by the dominated convergence theorem.

For $k = 1$, we can expand $u(x) = u(0) + xv(x),\ v \in C(\mathbb{R})$.
For $k = 2$, we can expand $u(x) = u(0) + xu'(0) + x^2 v(x),\ v \in C(\mathbb{R})$.

Problem 1.2

Test functions $u \in C_0^\infty(\mathbb{R})$.

1. $\langle \mathrm{pv}\frac{1}{x}, u\rangle = \lim_{\epsilon\to 0} \int\limits_{|x|\geq\epsilon} \frac{u(x)}{x}\,dx$. Is $\mathrm{pv}\frac{1}{x}$ a distribution?

2. $\langle \mathrm{fp}\frac{1}{x^2}, u\rangle = \lim_{\epsilon\to 0} \left(\int\limits_{|x|\geq\epsilon} \frac{u(x)}{x^2} - 2\frac{u(0)}{\epsilon} \right)$. Is it a distribution?

3. $\langle \mathrm{fp}\frac{H}{x^2}, u\rangle = \lim_{\epsilon\to 0} \left(\int\limits_{|x|\geq\epsilon} \frac{u(x)}{x^2} - \frac{u(0)}{\epsilon} + u'(0)\log\epsilon \right)$. Is it a distribution?

$$\begin{aligned}\left\langle \mathrm{pv}\frac{1}{x}, u\right\rangle &= \lim_{\epsilon\to 0}\int_{M\geq|x|\geq\epsilon} \frac{u(0)}{x} + v(x)\,dx \\ &= \lim_{\epsilon\to 0}\int v(x)\,dx + \underbrace{u(0)\lim_{\epsilon\to 0}\int \frac{1}{x}\,dx}_{0(\text{odd function})}\end{aligned}$$

$$\left|\left\langle \mathrm{pv}\frac{1}{x}, u\right\rangle\right| \leq C\sup|v(x)| \leq C\sup|u'(x)|$$

Problem 1.3

Define δ such that $\delta * f = f \;\forall\; f \in C_0^0(\mathbb{R})$. Prove that there does not exist $\delta \in C_0^0(\mathbb{R})$ satisfying $\delta * f = f$. If there was such a δ, then $f(x) = \int_{\mathbb{R}} \delta(x-y)f(y)\,dy$. Hint: $f(0) = \int_{\mathbb{R}} \delta(-y)f(y)\,dy$.

Use tent functions f_n with $f(-1/n) = f(1/n) = 0$ and $f(0) = 2n$.

$$\begin{aligned}
|f_n(0)| &\leq \|\delta\|_{L^2}\|f_n\|_{L^2} \\
\|f_n\|_{L^2} &= 1 \\
n &\leq |f_n(0)| \leq \|\delta\|_{L^2} \\
f_n(0) &\geq n \\
\|\delta\|_{L^2} &\geq n \;\forall\; n
\end{aligned}$$

From Shkoller: use the fact that the integral of f_n is 1 and $f_n(0) = n = \int_{-1/n}^{1/n} \delta(-y)f_n(y)\,dy$. Thus

$$\int_{-1/n}^{1/n} (n - \delta(y))f_n(y)\,dy = 0.$$

Or you can use a rectangular function with height n and base $[-1/n, 1/n]$. There exists $\delta \in L^1$

Problem 1.4

Poincaré Inequality. Let $u \in C_0^\infty(\mathbb{R}^n)$.

1. $\int_{\mathbb{R}^n} |u(x)|^2\,dx = -C_n \int_{\mathbb{R}^n} Du \cdot xu\,dx$. Find C_n.
2. $\int_\Omega |u(x)|^2\,dx \le C \int_\Omega |Du(x)|^2\,dx \;\forall\; u \in C_0^\infty(\Omega),\ \Omega \subset \mathbb{R}^n$ bounded

1. In 1-D:

$$\begin{aligned} -\int_{-\infty}^{\infty} \frac{du}{dx} ux\,dx &= -\int_{-\infty}^{\infty} \frac{1}{2}\frac{d}{dx}|u|^2 x\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2}|u|^2\,dx - \cancel{\frac{1}{2}|u|^2 x\Big|_{-\infty}^{\infty}} \end{aligned}$$

In n dimensions:

$$\begin{aligned} -\int_{\mathbb{R}^n} Du \cdot ux\,dx &= \lim_{M\to\infty} \int_{B(0,M)} Du \cdot ux\,dx \\ &= -\lim_{M\to\infty} \int_{B(0,M)} \frac{1}{2} D|u|^2 \cdot x\,dx \\ D|u|^2 &= D\left(\sum_{i=1}^n u_i u_i\right) \\ -\int_{\mathbb{R}^n} Du \cdot ux\,dx &= \lim_{M\to\infty}\left(\cancel{-\int_{\partial B(0,M)} \frac{1}{2}|u|^2 x \cdot n\,dS} + \frac{n}{2}\int_{B(0,M)} |u|^2\,dx\right) \\ \int_{\mathbb{R}^n} |u(x)|^2\,dx &= -\frac{2}{n}\int_{\mathbb{R}^n} Du \cdot xu\,dx \end{aligned}$$

2. Use part 1 (and Cauchy-Schwarz).

$$\begin{aligned} \int_\Omega |u|^2\,dx = \|u\|_{L^2}^2 &\le C\|Du\|_{L^2}\|u\|_{L^2} \\ \|u\|_{L^2}^2 &\le C^2\|Du\|_{L^2}^2 \end{aligned}$$

Problem 1.5

Singular integrals. Suppose $u \in C^\infty(\mathbb{R}^n - \{0\})$, with $u(rx) = r^{-n}u(x)$. Prove that

$$\langle T, \varphi \rangle = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} u(x)\varphi(x)\, dx$$

exists for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ if and only if

$$\int_{S^{n-1}} u(\omega)\, d\omega = 0.$$

Let's use $n = 2$. Hint: Taylor expand φ (1st order).

$$\int_\epsilon^M \int_0^{2\pi} u(r,\theta)\varphi(r,\theta) r\, dr\, d\theta = \int_\epsilon^M \int_0^{2\pi} \frac{1}{r} u(1,\theta)\varphi(r,\theta)\, d\theta\, dr$$

How can we get rid of this non-integrability? Expand φ. So we get to something like

$$\int_\epsilon^M \frac{1}{r} \int_0^{2\pi} u(1,\theta)\, d\theta\, dr$$

Problem 1.6

Compute $x^2\delta'$ and $x\delta''$ as distributions, where δ is the delta distribution.

$$\begin{aligned}\left\langle x^2\delta', \varphi\right\rangle &= \left\langle \delta', x^2\varphi\right\rangle \\ &= -\left\langle \delta, (x^2\varphi)'\right\rangle \\ &= -\left\langle \delta, x^2\varphi' + 2x\varphi\right\rangle \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle x\delta'', \varphi\rangle &= \langle \delta'', x\varphi\rangle \\ &= \langle \delta, (x\varphi)''\rangle \\ &= \langle \delta, x\varphi'' + 2\varphi'\rangle \\ &= 2\varphi'(0)\end{aligned}$$

Problem 1.7

Compute $e^{ax}\delta'$ in $\mathcal{S}'$.

$$
\begin{aligned}
\langle e^{ax}\delta', \varphi \rangle &= \langle \delta', e^{ax}\varphi \rangle \\
&= -\left\langle \delta, \frac{d}{dx}\left(e^{ax}\varphi\right) \right\rangle \\
&= -\left\langle \delta, a e^{ax}\varphi + e^{ax}\varphi' \right\rangle \\
&= -a\varphi(0) - \varphi'(0)
\end{aligned}
$$

Problem 1.8

True or False. $T \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

(a) $\langle T, \varphi \rangle = 0$ implies $\varphi T = 0$?

(b) $\varphi T = 0$ implies $\langle T, \varphi \rangle = 0$?

(a) Choose

$$\varphi = e^{-x^2}, \qquad \psi = e^{-(x+1)^2}.$$

$$\begin{aligned}
\varphi\psi &= e^{-x^2} e^{-(x+1)^2} \\
&= e^{-2x^2-2x-1} \\
\langle \delta', \varphi\psi \rangle &= \left\langle \delta', e^{-2x^2-2x-1} \right\rangle \\
&= \left\langle \delta, (-4x-2)e^{-2x^2-2x-1} \right\rangle \\
&= -2e^{-1} \\
&\neq 0
\end{aligned}$$

Problem 1.9

Compute the distributional derivative of $\log|x|$ on $\mathbb{R}$.

Since $\log|x| \in L^1_{\text{loc}}$, i.e. $\log|x|$ is locally integrable, we write

$$\begin{aligned}
\langle (\log|x|)', \varphi \rangle &= -\langle \log|x|, \varphi' \rangle \\
&= -\int_{\mathbb{R}} \log|x|\varphi'(x)\,dx \\
&= -\int_{|x|\le\epsilon} \log|x|\varphi'(x)\,dx - \int_{|x|\ge\epsilon} \log|x|\varphi'(x)\,dx \\
\int_{|x|\ge\epsilon} \log|x|\varphi'(x)\,dx &= \log|x|\varphi(x)\big|_{-\epsilon}^{\epsilon} - \int_{|x|\ge\epsilon} \frac{\varphi(x)}{x}\,dx \\
&= \log\epsilon\,\underbrace{(\varphi(\epsilon)-\varphi(-\epsilon))}_{\le|\varphi'(0)|2\epsilon\to 0} - \int_{|x|\ge\epsilon} \frac{\varphi(x)}{x}\,dx && \text{(Mean Value Theorem)} \\
\int_{|x|\le\epsilon} \log|x|\varphi'(x)\,dx &= \int_{|x|\le\epsilon} \log|x|(\varphi'(x)-\varphi'(0))\,dx + \int_{|x|\le\epsilon} \log|x|\varphi'(0)\,dx \\
\mathbf{1}_{|x|\ge\epsilon}\log|x|\varphi' &\le \log|x|\varphi' \in L^1 && \text{(LDCT)}
\end{aligned}$$

Or...

$$\begin{aligned}
\langle (\log|x|)', \varphi \rangle &= -\langle \log|x|, \varphi' \rangle \\
&= -\int_{\mathbb{R}} \log|x|\varphi'(x)\,dx \\
&= -\int_{\mathbb{R}} \lim_{\epsilon\to 0} \mathbf{1}_{|x|\ge\epsilon}\log|x|\varphi'(x)\,dx \\
&= \lim_{\epsilon\to 0}\int_{|x|\ge\epsilon} \log|x|\varphi'(x)\,dx
\end{aligned}$$

I think an integration by parts step has been omitted here. . .

$$\frac{d}{dx}\log|x| = \text{pv}\frac{1}{x} \quad \text{in } \mathcal{D}'$$

Problem 1.10

Compute the distributional derivative of $\mathrm{pv}\frac{1}{x}$ in $\mathcal{S}'(\mathbb{R})$. Hint: Taylor expand $\epsilon\varphi(\epsilon)$ and $\epsilon\varphi(-\epsilon)$.

$$\frac{d}{dx}\mathrm{pv}\frac{1}{x} = -\mathrm{fp}\frac{1}{x^2} \quad \text{in } \mathcal{S}'$$

Problem 1.11

Compute $\mathcal{F}\left(\text{pv}\frac{1}{x}\right), \mathcal{F}(H)$. H is the Heaviside function:

$$H = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

1. By the LDCT, $\text{pv}\frac{1}{x} \cdot x = 1$ in $\mathcal{S}'(\mathbb{R})$.
2.

$$\begin{aligned}
\mathcal{F}(xf(x)) &= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \underbrace{(-i)xe^{-ix\xi}}_{\frac{d}{d\xi}e^{-ix\xi}} \, dx \\
&= \frac{i}{\sqrt{2\pi}} \frac{d}{d\xi} \int_{\mathbb{R}} f(x)e^{-ix\xi} \, dx \\
&= i\frac{d}{d\xi}\hat{f}(\xi)
\end{aligned}$$

Putting (1) and (2) together, we get that

$$\begin{aligned}
\mathcal{F}\left(x \cdot \text{pv}\frac{1}{x}\right) = \mathcal{F}(1) &= \frac{1}{\sqrt{2\pi}}\delta \\
&= \frac{d}{d\xi}\hat{f}(\xi) \\
&= \frac{1}{i\sqrt{2\pi}}\delta
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(1) &= \frac{1}{\sqrt{2\pi}}\delta \\
\mathcal{F}^*\mathcal{F}(\delta) &= \mathcal{F}^*\frac{1}{\sqrt{2\pi}}
\end{aligned}$$

We have $\hat{f}(\xi) = \frac{1}{i\sqrt{2\pi}}H + C$. This must be odd.

$$\begin{aligned}
\frac{1}{i\sqrt{2\pi}} + C &= -C \\
C &= \frac{i}{2\sqrt{2\pi}}
\end{aligned}$$

So

$$\mathcal{F}\left(\text{pv}\frac{1}{x}\right) = \frac{H}{i\sqrt{2\pi}} + \frac{i}{2\sqrt{2\pi}}.$$

$$\begin{aligned} H &= i\sqrt{2\pi}\mathcal{F}\left(\text{pv}\frac{1}{x}\right) + \frac{1}{2} \\ &= -i\sqrt{2\pi}\mathcal{F}^*\left(\text{pv}\frac{1}{x}\right) + \frac{1}{2} \\ \mathcal{F}H &= i\sqrt{2\pi}\text{pv}\frac{1}{x} + \mathcal{F}\left(\frac{1}{2}\right) \\ &= i\sqrt{2\pi}\text{pv}\frac{1}{x} + \frac{1}{2\sqrt{2\pi}}\delta \end{aligned}$$

Problem 1.12

T is even if $\langle T, \varphi(x)\rangle = \langle T, \varphi(-x)\rangle$. T is odd if $\langle T, \varphi(x)\rangle = -\langle T, \varphi(-x)\rangle$. **Prove.** For an even function, $\mathcal{F}T = \mathcal{F}^*T$. For an odd function, $\mathcal{F}T = -\mathcal{F}^*T$.

Even:

$$\begin{aligned}
\langle \mathcal{F}^*T, \varphi(x)\rangle &= \langle T, \mathcal{F}^*\varphi(x)\rangle \\
&= \left\langle T, (2\pi)^{-n/2} \int \varphi(x) e^{ik\cdot x}\, dx \right\rangle \\
&= \left\langle T, (2\pi)^{-n/2} \int \varphi(-x) e^{-ik\cdot x}\, dx \right\rangle \qquad (1.1) \\
&= \langle T, \mathcal{F}\varphi(-x)\rangle \\
&= \langle \mathcal{F}T, \varphi(-x)\rangle \\
&= \langle \mathcal{F}T, \varphi(x)\rangle \qquad (1.2)
\end{aligned}$$

There are 2 negative signs in (1.1) that cancel each other out, one because $d(-x) = -dx$ and the other from changing the order of integration.

Problem 1.13

Use the identity

$$|x| = xH(x) - xH(-x)$$

and what we computed for $\mathcal{F}H$, $\mathcal{F}^*H$, and $\frac{d}{d\xi}\widehat{\mathrm{pv}\frac{1}{x}}$ to compute $\mathcal{F}(x)$.

$$\begin{aligned}
\eta(x) &= H(x) - \frac{1}{2} \\
&= -\eta(-x) \\
&= -H(-x) + \frac{1}{2} \\
&= H(x) - \frac{1}{2} \\
H(x) &= \eta(x) + \frac{1}{2} \\
H(-x) &= \eta(-x) + \frac{1}{2}
\end{aligned}$$

9-13-11

Problem 2.1

Prove
$$\int_0^{\pi} x^{-1/4} \sin x \, dx \leq \pi^{3/4}$$

Hint: use Cauchy-Schwarz.

Problem 2.2

$f \in L^2(0, \pi)$. Is it possible that

$$\int_0^\pi [f(x) - \sin(x)]^2 \, dx \leq \frac{4}{9}$$
$$\int_0^\pi [f(x) - \cos(x)]^2 \, dx \leq \frac{1}{9}$$

For any $F, G \in L^2$, Minkowski's Inequality gives us that

$$\|F + G\|_{L^2} \leq \|F\|_{L^2} + \|G\|_{L^2}.$$

Let $F = f(x) - \sin x, \; G = f(x) - \cos x$.

$$\begin{aligned}
\left(\int_0^\pi \left[(f(x) - \sin x) - (f(x) - \cos x) \right]^2 \, dx \right)^{1/2} &= \left(\int_0^\pi (\sin x - \cos x)^2 \, dx \right)^{1/2} = \sqrt{\pi} \\
&\leq \left(\int_0^\pi [f(x) - \sin x]^2 \, dx \right)^{1/2} + \left(\int_0^\pi [f(x) - \cos x]^2 \, dx \right)^{1/2} \\
&\leq \frac{2}{3} + \frac{1}{3} \\
&= 1
\end{aligned}$$

Problem 2.3

Let $1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1,\ f \in L^p(\mathbb{R}),\ g \in L^q(\mathbb{R})$. Show that $f * g$ is continuous and that

$$\lim_{|x| \to \infty} (f * g)(x) = 0.$$

Hint: f and g can be approximated by "nice" functions.

Step 1: assume that $f, g \in C_c^\infty(\mathbb{R})$.

$$\begin{aligned}
\lim_{\delta \to 0} |f * g(x+\delta) - f * g(x)| &\leq \lim_{\delta \to 0} \int_{\mathbb{R}} |f(x-y+\delta) - f(x-y)||g(y)|\, dy \\
&\leq \int_{\mathbb{R}} \lim_{\delta \to 0} |f(x-y+\delta) - f(x-y)||g(y)|\, dy \qquad \text{(DCT)} \\
&= 0
\end{aligned}$$

Step 2: let f, g be the functions originally specified. There exist sequences $(f_n), (g_n) \subset C_c^\infty(\mathbb{R})$ such that $f_n \to f$ in L^p and $g_n \to g$ in L^q uniformly.

$$\begin{aligned}
\|f * g - f_n * g_n\|_\infty &= \|f * g - f * g_n + f * g_n - f_n * g_n\|_\infty \\
&\leq \|f * (g - g_n)\|_\infty + \|(f - f_n) * g\|_\infty \\
&\leq \|f\|_p \|g - g_n\|_q + \|g_n\|_q \|f - f_n\|_p \qquad \text{(Young's Inequality)} \\
&\to 0
\end{aligned}$$

The conclusions follow from this work.

Problem 2.4

Let $f \in L^p(0,1),\ p > 0$. This implies $f \in L^q(0,1)$ for $0 < q \leq p$. Prove

$$\lim_{q \to 0} \|f\|_{L^q} = \exp \int_0^1 \log |f| \, dx.$$

Hint: expand the norm as

$$\lim_{q \to 0} \left(\int_0^1 |f|^q \, dx \right)^{1/q}.$$

Keep in mind L'Hospital's Rule ($\frac{d}{dx} a^x$).

$$\begin{aligned}
\lim_{q \to 0} \log \left(\int_0^1 |f|^q \, dx \right)^{1/q} &= \lim_{q \to 0} \frac{1}{q} \log \left(\int_0^1 |f|^q \, dx \right) \\
&= \lim_{q \to 0} \frac{1}{\int_0^1 |f|^q \, dx} \cdot \left(\int_0^1 |f|^q \log |f| \, dx \right) && \text{(L'Hospital's)} \\
&= \int_0^1 \log |f| \, dx && \text{(DCT)}
\end{aligned}$$

From Shkoller:
We are given $f \in L^p,\ 0 < q \leq p$. We reduced it to this:

$$\lim_{q \to 0} \frac{\int |f|^q \log |f| \, dx}{\int |f|^q \, dx}$$

What is the dominating function for the top, specifically for when $|f| < 1$? "When you need to construct a dominating function for log, it's going to be tricky." This is a monotone sequence (in q), so think monotone convergence theorem.

Problem 2.5

Let $\Omega \subset \mathbb{R}^n$ bounded. $\Omega = (0,1)$ is OK.

$$\begin{aligned} \|f\|_{L^p(\Omega)} &\le Cp, \qquad p \ge 1,\ C \text{ independent of } p \\ f &\ge 0 \text{ a.e.} \end{aligned}$$

Show there exists $\tilde{c}$ such that $e^{\tilde{c}f} \in L^1(\Omega)$. Hint: use Stirling's formula to relate p^p to $p!$.

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

$$\begin{aligned} \int_0^1 e^{\tilde{c}f}\,dx &= \int_0^1 1 + \tilde{c}f + \frac{\tilde{c}^2 f^2}{2!} + \cdots\,dx \\ &= \sum_{n=0}^{\infty} \frac{\tilde{c}^n}{n!} \int_0^1 f^n\,dx && \text{(MCT)} \\ &\le \sum_{n=0}^{\infty} \frac{\tilde{c}^n}{n!} C^n n^n \\ &= \sum_{n=0}^{\infty} \frac{n^{n+\frac{1}{2}} (C\tilde{c})^n}{n!\sqrt{n}} \\ &= \sum_{n=0}^{\infty} e^n (C\tilde{c})^n \\ &= \sum_{n=0}^{\infty} (eC\tilde{c})^n \end{aligned}$$

Problem 2.6

Let $f \in L^1(\mathbb{R})$ and $h > 0$. Set

$$g_h(x) = \frac{1}{2h} \int_x^{x+h} f(y)\, dy.$$

Thus, $g_h \in L^1(\mathbb{R})$. Show that

$$\int_{\mathbb{R}} |g_h(x)|\, dx \leq \|f\|_{L^1(\mathbb{R})}.$$

$$\begin{aligned}
\|g_h\|_{L^1} &= \int_{\mathbb{R}} |g_h(x)|\, dx \\
&= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_x^{x+h} f(y)\, dy \right| dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2h} \int_0^h |f(x+y)|\, dy\, dx \\
&= \lim_{M\to\infty} \int_{-M}^{M} \frac{1}{2h} \int_0^h |f(x+y)|\, dy\, dx \\
&= \lim_{M\to\infty} \frac{1}{2h} \int_0^h \int_{-M}^{M} |f(x+y)|\, dx\, dy \\
&\leq \lim_{M\to\infty} \frac{1}{2h} \int_0^h \|f\|_{L^1(\mathbb{R})} \\
&= \frac{1}{2} \|f\|_{L^1(\mathbb{R})}
\end{aligned}$$

From Shkoller:

$$\frac{1}{2h} \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{\mathbf{1}_{[x,x+h]}(y)}_{=\mathbf{1}_{[y-h,y]}(x)} f(y)\, dy\, dx$$

Problem 2.7

Let $f \in L^1(\mathbb{R})$ and we know that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(4x) f(x+y)\, dx\, dy = 1.$$

Compute $\int_{\mathbb{R}} f(x)\, dx$.

By Tonelli's Theorem, change the order of integration:

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} f(4x) f(x+y)\, dx\, dy &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(4x) f(x+y)\, dy\, dx \\
&= \int_{\mathbb{R}} f(4x)\, dx \int_{\mathbb{R}} f(x+y)\, dy \\
&= \frac{1}{4} \int_{\mathbb{R}} f(x)\, dx \int_{\mathbb{R}} f(x)\, dx = 1 \\
\int_{\mathbb{R}} f(x)\, dx &= \pm 2
\end{aligned}$$

We used the translation invariance of Lebesgue measure.

Problem 2.8

$(f_n) \subset L^p(\Omega)$, Ω is bounded.

1. $\|f_n\|_{L^p(\Omega)} \leq C$
2. $f_n \to f$ a.e.

Prove $f_n \to f$ in $L^q(\Omega)$ for all $1 \leq q < p$.

$$\begin{aligned}
\|f_n - f\|^q_{L^q(\Omega)} &= \|f_n - f\|^q_{L^q(E)} + \|f_n - f\|^q_{q(E^c)} \\
&\leq \|f_n - f\|^q_{L^q(E)} + \epsilon\mu(E^c) \\
&\leq \int_E |f - f_n|^q \, dx + \epsilon\mu(E^c) \\
&\leq \|f - f_n\|^q_{L^p} \mu(E)^{1-q/p} + \epsilon\mu(E^c) && \text{(Hölder's)} \\
&\leq (2c)^q \delta^{1-q/p} + \epsilon\mu(E^c)
\end{aligned}$$

Problem 2.9

$f_n, f \in L^1$, $f_n \to f$ a.e. and $\|f_n\|_{L^1} \to \|f\|_{L^1}$. Prove $\|f_n - f\|_{L^1} \to 0$. Hint: use Fatou's Lemma.

Consider the sequence

$$g_n = |f| + |f_n| - |f_n - f| \geq 0.$$

Then

$$\begin{aligned} 2\int_\Omega |f|\,dx &= \int_\Omega \lim_{n\to\infty} g_n\,dx \\ &\leq \liminf_{n\to\infty} \int_\Omega \left(|f| + |f_n| - |f_n - f|\right)\,dx \\ &\leq 2\int_\Omega |f|\,dx - \limsup_{n\to\infty} \int_\Omega |f_n - f|\,dx \end{aligned}$$

If this was L^p instead of L^1 then it would still be true, except we would need to be clever about our sequence $g_n \Rightarrow$ for $n = 2$, set $g_n = |f|^2 + |f_n|^2 - |f_n - f|^2$.

Now suppose $f_n \rightharpoonup f$ in L^1. Is it still true?

No.

Problem 2.10

Construct a sequence $(f_n) \subset L^1(I)$, $f_n \geq 0$, such that

1. $f_n \rightharpoonup f$ in L^1
2. $\|f_n\|_{L^1} \to \|f\|$
3. $\|f_n - f\|_{L^1} \not\to 0$

Shkoller's suggestion:

$$f_n = 1 + \sin(nx)$$

Let $I = (0, 2\pi)$.

1. $f_n \rightharpoonup 1$, since $\sin(nx) \rightharpoonup 0$ via oscillation
2. $\|f_n\| \to 2\pi = \|1\|$
3.

Problem 2.11

For $L^p,\ 1 < p < \infty$, do the conditions of the previous problem give us strong convergence?

Problem 2.12

Let $au \in L^q(\Omega)\ \forall\ u \in L^p(\Omega),\ 1 \leq q \leq p \leq \infty$. Show that $a \in L^r(\Omega)$, where

$$r = \begin{cases} \frac{pq}{p-q} & p < \infty \\ q & p = \infty \end{cases}$$

Problem 2.13

Let $f \in L^1(a,b)$ and $\int\limits_a^c f(y)\,dy = 0$ for all $c \in [a,b]$. Prove that $f = 0$ a.e.

Note that

$$\int_a^c f(y)\,dy = \int_a^b \mathbf{1}_{[a,c]} f(y)\,dy = 0.$$

For any $\varphi \in C^\infty(a,b)$, we can approximate φ by simple functions. Also note that $\mathbf{1}_{[j,k]} = \mathbf{1}_{[a,k]} - \mathbf{1}_{[a,j]}$. Thus, for any simple function φ_{sim} we have that

$$\int_a^b \varphi_{\text{sim}} f(y)\,dy = 0.$$

Therefore, for any $\varphi \in C^\infty(a,b)$ we will have that

$$\int_a^b \varphi_{\text{sim}} f(y)\,dy = 0.$$

We have proven before that this implies $f = 0$ a.e.

9-14-11

Problem 3.1

Is $\delta \in \mathcal{H}^{-1}(\mathbb{R})$? Recall that $\mathcal{H}^{-1}(\mathbb{R}) = [\mathcal{H}^1(\mathbb{R})]'$.

In order for δ to be well-defined, we need continuous functions. Are the functions in $\mathcal{H}^1(\mathbb{R})$ continuous? Yes. So $\delta \in \mathcal{H}^{-1}(\mathbb{R})$.

Sobolev embedding:

$$\max_{x\in\Omega} |u(x)| \leq c\|u\|_{\mathcal{H}^s(\Omega)}, \qquad s > \frac{n}{2},\ \Omega \subset \mathbb{R}^n$$

The Riesz Representation Theorem tells us that there exists $u \in \mathcal{H}^1(\mathbb{R})$ such that

$$(u, \varphi)_{\mathcal{H}^1(\mathbb{R})} = \langle \delta, \varphi \rangle \qquad \forall\ \varphi \in \mathcal{H}^1(\mathbb{R})$$

Can we find this u? It satisfies

$$\begin{aligned}
\int_{\mathbb{R}} \left[u(x)\varphi(x) + \frac{du}{dx}(x)\frac{d\varphi}{dx}(x) \right] dx &= \langle \delta, \varphi \rangle = \varphi(0) \quad \forall\ \varphi \in \mathcal{H}^1(\mathbb{R}) \\
u - \frac{d^2u}{dx^2} &= \delta \quad \text{in } \mathcal{H}^{-1}(\mathbb{R}) \\
(1 - \xi^2)\hat{u}(\xi) &= \frac{1}{\sqrt{2\pi}} \\
\hat{u}(\xi) &= \frac{1}{\sqrt{2\pi}(1+\xi^2)} \\
u(x) &= \mathcal{F}^*(1 + 2\xi)/\sqrt{2\pi} \\
&= \frac{e^{-|x|}}{\sqrt{2\pi}}
\end{aligned}$$

Problem 3.2

Let $\Omega \subset \mathbb{R}^n$ smooth, $n = 2$, ψ is a C^∞ diffeomorphism of Ω, $u \in \mathcal{H}^k(\Omega)$, $k > \frac{n}{2} + 1$, $k = 3$. Prove that $u \circ \psi$ is also in $\mathcal{H}^k(\Omega)$.

To make this simpler, consider $u \in \mathcal{H}^3(\Omega) \cap \mathcal{H}^1_0(\Omega)$. Hint: for $u \in \mathcal{H}^3(\Omega) \cap \mathcal{H}^1_0(\Omega)$,

$$\|u\|_{H^3(\Omega)} = \|D^3 u\|_{L^2(\Omega)}$$

Further Hint: if $\|D^3 u\|_{L^2(\Omega)} \leq C$, then $\|D^3(u \circ \psi)\|_{L^2(\Omega)} \leq \tilde{C}$.

$u \in H^3(\Omega)$. Assume $u \in C_0^\infty(\Omega)$ (density argument).

$$D(u \circ \psi) = Du \circ \psi \cdot D\psi \qquad \text{(chain rule)}$$

$$\vdots$$

$$D^3(u \circ \psi) = D^3 u \circ \psi(D\psi, D\psi, D\psi) + Du \circ \psi D^3\psi + \text{l.o.t.}$$

$$\int_\Omega |D^3(u \circ \psi)|^2\, dx = \underbrace{\int_\Omega |D^3 u \circ \psi|^2 |(D\psi)|^6\, dx}_{\mathcal{I}} + \int_\Omega |Du \circ \psi|^2 |D^3\psi|^2\, dx + \int_\Omega \text{l.o.t.}$$

$$\begin{aligned} \mathcal{I} &\leq \|D\psi\|^6_{L^\infty} \int_\Omega |D^3 u \circ \psi|^2\, dx \\ &\leq \|D\psi\|^6_{L^\infty} \int_\Omega |D^3 u|^2 |\det D\psi|\, dx \\ &\leq \underbrace{\|D\psi\|^6_{L^\infty} \|\det D\psi\|_{L^\infty}}_{C} \|D^3 u\|^2_{L^2(\Omega)} \end{aligned}$$

Problem 3.3

Multiplicative Algebra. Suppose that $u \in \mathcal{H}^s(\mathbb{R}^n)$ and $v \in \mathcal{H}^s(\mathbb{R}^n)$, $s > \frac{n}{2}$. Prove that

$$\|uv\|_{\mathcal{H}^s(\mathbb{R}^n)} \le c\|u\|_{\mathcal{H}^s(\mathbb{R}^n)}\|v\|_{\mathcal{H}^s(\mathbb{R}^n)}$$

Hint: use Fourier Transform.

$$\|u\|^2_{\mathcal{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi$$

$$\begin{aligned}
\|uv\|_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |\widehat{uv}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |\hat{u} * \hat{v}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \left| \int_{\mathbb{R}^n} \hat{u}(y-\xi)\hat{v}(y)\, dy \right|^2 \, d\xi \\
&\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle\xi + y\rangle^{2s} |\hat{u}(\xi)|^2 |\hat{v}(y)|^2 \, dy \, d\xi \\
&= \int\int (1+|\xi|^2+|y|^2+2\xi y)^2 |\hat{u}(\xi)|^2 |\hat{v}(y)|^2 \, dy \, d\xi \\
&\le C \int\int \langle\xi\rangle^{2s} \langle y\rangle^{2s} |\hat{u}(\xi)|^2 |\hat{v}(y)|^2 \, dy \, d\xi \\
&= C\left(\int \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 \, d\xi\right)\left(\int \langle y\rangle^{2s} |\hat{v}(y)|^2 \, dy\right) \\
&= C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}
\end{aligned}$$

Proof that $\widehat{fg} = \hat{f} * \hat{g}$:
The convolution theorem gives us

$$\begin{aligned}
\widehat{f * g} &= \hat{f}\hat{g} \\
f \check{*} g &= \check{f}\check{g}.
\end{aligned}$$

Let $\hat{f} = F,\ \hat{g} = G$. Applying the 2nd version of the convolution theorem to these functions, we get

$$F \check{*} G = \check{F}\check{G}.$$

Applying the Fourier transform to this yields

$$F * G = \widehat{\check{F}\check{G}}.$$

Substituting f and g back in we see

$$\hat{f} * \hat{g} = \widehat{fg}.$$

From Shkoller:

Let's work in $\mathbb{R}$, so $n = 1$.

$$\begin{aligned}
\|uv\|_{H^1}^2 &\le c\|u\|_{H^1}^2\|v\|_{H^1}^2 \\
\int_{\mathbb{R}} u^2v^2\,dx + \int_{\mathbb{R}} (uv)'^2\,dx &= \int_{\mathbb{R}} u^2v^2\,dx + \int_{\mathbb{R}} (u'^2v^2 + 2uu'vv' + u^2v'^2)\,dx \\
&\overset{?}{\le} c\|u\|_{H^1(\mathbb{R})}^2\|v\|_{H^1(\mathbb{R})}^2 \\
\int_{\mathbb{R}} \left(\frac{du}{dx}\right)^2 v^2\,dx &\le \left\|\frac{du}{dx}\right\|_{L^2}^2 \|v\|_{L^\infty}^2 \\
&\le \|u\|_{H^1}^2\|v\|_{L^\infty}^2 \\
&\le C\|u\|_{H^1}^2\|v\|_{H^1}^2 && \text{(Sobolev Embedding)}
\end{aligned}$$

$$\begin{aligned}
\|uv\|_{H^1}^2 &= \int_{\mathbb{R}} (1+\xi^2)|\mathcal{F}(uv)|^2\,d\xi \\
&= \int_{\mathbb{R}} (1+\xi^2)|\hat{u} * \hat{v}|^2\,d\xi \\
&= \underbrace{\int_{\mathbb{R}} |\hat{u} * \hat{v}|^2\,d\xi}_{\|\hat{u}*\hat{v}\|_{L^2}^2} + \int_{\mathbb{R}} \xi^2|\hat{u} * \hat{v}|^2\,d\xi \\
&\le \underbrace{\frac{1}{2}\left(\|hatu\|_{L^2}^2\|\hat{v}\|_{L^1}^2 + \|\hat{v}\|_{L^2}^2\|hatu\|_{L^1}^2\right)}_{\text{Young's Inequality}} + \int_{\mathbb{R}} \xi^2|\hat{u} * \hat{v}|^2\,d\xi
\end{aligned}$$

Consider

$$\int \xi\hat{u}(\xi-\eta)\hat{v}(\eta)\,d\eta.$$

Use a change of variables $(\xi - \eta + \eta)$ to move the ξ into the convolution. Then we get something like

$$\|(\xi\hat{u}) * \hat{v}\|_{L^2}^2 \le \|\xi\hat{u}\|_{L^2}^2\|\hat{v}\|_{L^1}^2$$

The L^1 norm of the Fourier transform corresponds to the L^∞ norm of the original function.

Problem 3.4

Suppose $u \in C_0^\infty(\mathbb{R}^3)$. By Gagliardo-Nirenberg,

$$\alpha\|Du\|_{L^2} \geq \beta\|u\|_{L^6}$$

Now suppose $u \in C_0^\infty(\mathbb{R}^2)$. By the second Poincaré Inequality,

$$C\sqrt{q}\|Du\|_{L^2} \geq \|u\|_{L^q}, \qquad 1 \leq q < \infty \tag{3.1}$$

Now consider the 2-D domains:

$$\Omega_1 = \{(x,y) \mid 0 < x < 1; 0 < y < x\}$$
$$\Omega_2 = \{(x,y) \mid 0 < x < 1; 0 < y < x^2\}$$
$$\Omega_\beta = \{(x,y) \mid 0 < x < 1; 0 < y < x^\beta\}$$

Consider $u(x) = |x|^\alpha$. Take $\beta = 2$. For $u \in L^8(\Omega_2)$, find α such that $u \in \mathcal{H}^1(\Omega_2)$.

First consider $u(x) = x_1^\alpha$.

For $u(x) = x_1^\alpha$, $u \in L^8(\Omega_2)$ if $\alpha > -\frac{3}{8}$ and $u \in H^1(\Omega_2)$ if $\alpha > -\frac{1}{2}$. Thus, the curvature of the domain ruins the embedding in 3.1.

Problem 3.5

Let $u \in L^2(\mathbb{R})$, with spt u compact. Let

$$\eta_\epsilon(x) = \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right), \qquad \text{spt } \eta_\epsilon \subset \overline{B(0,\epsilon)}, \quad \eta_\epsilon \in C_0^\infty$$
$$u_\epsilon = \eta_\epsilon * u$$
$$\left\|\frac{du_\epsilon}{dx}\right\|_{L^2(\mathbb{R})} \leq M.$$

Prove that $u \in \mathcal{H}^1(\mathbb{R})$.

The sequence is bounded, so by Banach-Alaoglu we have a weakly convergent subsequence: $\frac{du_\epsilon}{dx} \rightharpoonup g$. We need to show g is bounded and that $g = Du$.

1. $\|g\|_{L^2} \leq \liminf_{n\to\infty} \left\|\frac{du_\epsilon}{dx}\right\| \leq M$.
2. Need
$$\int g\varphi\, dx = -\int u\varphi'\, dx \quad \forall\ \varphi \in C_0^\infty$$

$$\begin{aligned} \int g\varphi\, dx &= \lim_{\epsilon\to 0}\int u_\epsilon'\varphi\, dx \\ &= \lim_{\epsilon\to 0} -\int u_\epsilon\varphi'\, dx \\ &= -\int u\varphi'\, dx \end{aligned}$$

Remarks from Shkoller:
If $(u_\epsilon) \subset H^1$ and $\|u_\epsilon\|_{H^1(0,1)} \leq M$, then

$$\begin{aligned} u_{\epsilon'} &\rightharpoonup u \text{ in } H^1(0,1) \\ u_{\epsilon'} &\to u \text{ in } L^2(0,1) \end{aligned}$$

If $\|u_\epsilon\|_{H^s(\mathbb{T}^n)} \leq M$, then

$$\begin{aligned} u_{\epsilon'} &\rightharpoonup u \text{ in } H^s(\mathbb{T}^n) \\ u_{\epsilon'} &\to u \text{ in } H^r(\mathbb{T}^n) \quad \forall\ r < s \end{aligned}$$

Problem 3.6

Consider $B(0,1) \subset \mathbb{R}^3$. Define $u(x) = \frac{1}{|x|}$. Show that $Du = -\frac{x}{|x|^3}$.

We need to show that for any $\varphi \in C_0^\infty(B(0,1))$,

$$\int -\frac{x}{|x|^3}\varphi\,dx = -\int \frac{1}{|x|}\frac{\partial}{\partial x_i}\varphi\,dx,\ \ i = 1,2,3.$$

We split the integral up as

$$\begin{aligned}
\int_{B(0,1)-B(0,\epsilon)} \frac{1}{|x|}\frac{\partial}{\partial x_i}\varphi\,dx &= \int_{\partial B(0,\epsilon)} \frac{1}{|x|}\varphi\mathbf{n}_i\,dS - \int_{B(0,1)-B(0,\epsilon)} \varphi\left(-\frac{x_i}{|x|^3}\right)\,dx \\
&= \cancel{\int_{\partial B(0,1)} \frac{1}{|\epsilon|}\varphi\mathbf{n}_i\epsilon^2\,d\omega} - \int_{B(0,1)-B(0,\epsilon)} \varphi\left(-\frac{x_i}{|x|^3}\right)\,dx \\
&= -\int_{B(0,1)} \varphi\left(-\frac{x}{|x|^3}\right)\,dx
\end{aligned}$$

Problem 3.7

$$u_j \rightharpoonup u \text{ in } W_0^{1,1}(0,1)$$
$$\frac{du_j}{dx} \rightharpoonup \frac{du}{dx} \text{ in } L^1(0,1)$$

Prove $u_j \to u$ a.e.

$$\begin{aligned}
u_j(x) &\overset{\text{FTC}}{=} \cancel{u_j(0)} + \int_0^x \frac{du_j}{dy}(y)\,dy \\
\lim_{j\to\infty} u_j(x) &= \lim_{j\to\infty} \int_0^x \frac{du_j}{dy}(y)\,dy \\
&= \int_0^1 \mathbf{1}_{[0,x]} \frac{du_j}{dy}(y)\,dy \\
&\overset{\text{DCT}}{=} \int_0^x \lim_{j\to\infty} \mathbf{1}_{[0,x]} \frac{du_j}{dy}(y)\,dy \\
&= \int_0^x \frac{du}{dy}(y)\,dy \\
&= u(x) \quad \text{a.e.}
\end{aligned}$$

Problem 3.8

If (u_j) is bounded in $H^1(0,1)$, what does $u_j \frac{du_j}{dx}$ converge weakly to in $L^2(0,1)$ and why?

We know that there exists M such that

$$\|u_j\|_{L^2}, \left\|\frac{du_j}{dx}\right\|_{L^2} < M.$$

Then by Banach-Alaoglu, $u_j \rightharpoonup g$ and $\frac{du_j}{dx} \rightharpoonup h$.

$$\int_0^1 u_j \frac{du_j}{dx}\phi\,dx \to \int_0^1 u\frac{du}{dx}\phi\,dx \quad \forall\ \phi \in L^2(0,1)$$

$$\int (u_j - u)\frac{du_j}{dx}\phi\,dx + \int \underbrace{\left(\frac{du_j}{dx} - \frac{du}{dx}\right)}_{\to 0 \text{ weakly}} \overbrace{(\underbrace{\phi}_{L^2}\ \underbrace{u}_{L^\infty})}^{\text{want } L^2} \leq \left\|\frac{du_j}{dx}\right\|_{L^2} \|\phi\|_{L^2}$$

Problem 3.9

Let $u : \mathbb{R} \to \mathbb{R}$,

$$\hat{u}(\xi) = \frac{1}{\langle \xi \rangle \left(1 + \log \langle \xi \rangle\right)}, \qquad \langle \xi \rangle = \sqrt{1 + \xi^2}$$

Suppose $u \in H^{1/2}(\mathbb{R})$ but $u \notin L^\infty$. Thus, $\mathcal{F}u \notin L^1(\mathbb{R})$ (since $\int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx$). Note that $|\xi| < \langle \xi \rangle$. Prove that

$$\|u\|^2_{H^{1/2}(\mathbb{R})} \le C + 2 \int_1^\infty \frac{1}{|\xi|(1 + |\xi|^2)}\, d\xi$$

Hint: apply the change of variables $s = 1 + \log |\xi|$.

See 201C Practice Final #1 Problem 2.

9-15-11

Problem 4.1

$f \in L^1([a,b])$ and

$$\int_a^c f(x)\,dy = 0 \quad \forall\ c \in [a,b]$$

Prove that this implies $f = 0$ a.e. This relies on the Lebesgue differentiation theorem:

$$f(x) = \lim_{\delta \to 0} \int_x^{x+\delta} f(y)\,dy \quad \text{a.e.}$$

The Lebesgue differentiation theorem is used to prove the fundamental theorem of calculus.

$$\begin{aligned} g(c) &= \int_a^c f(y)\,dy = 0 \\ g'(c) &= 0 \\ f(c) &= 0 \quad \text{a.e. by FTOC} \end{aligned}$$

The following statement is nonsense (set $c = 0$):

$$\int_a^c f(y)\,dy = \beta \neq 0$$

Problem 4.2

Is $L^\infty(\mathbb{R})$ a Hilbert space?

No. The parallelogram law states that

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 2\|f\|_\infty^2 + \|g\|_\infty^2$$

Choose f and g with disjoint support. For example, $f = \mathbf{1}_{(-\infty,0]},\ g = \mathbf{1}_{[0,\infty)}$.

Problem 4.3

Let M be a closed linear subspace of a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ but $x \notin M$. Prove that there is a unique point in M minimizing the distance $(x, y)_{y \in M}$.

See H&N page 131.

Problem 4.4

$F : \mathbb{R}^n \to \mathbb{R}$ convex and C^1. $K \subset \mathbb{R}^n$ convex $u \in \mathbb{R}^n$. Show that $F(u) \leq F(v)\ \forall\ v \in K$ implies $(F'(u), v-u) \geq 0\ \forall\ v \in K$.

$$\begin{aligned}
F(v) &= F(u) + DF(u)\cdot(v-u) + O(\|v-u\|^2) \\
F(u) &\leq F(v) \\
DF(u)\cdot(v-u) &\geq 0
\end{aligned}$$

From Shkoller (to make it rigorous):
Convexity says that

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v).$$

We know that

$$\begin{aligned}
F(u) &\leq F(v) \\
tF(u) &\leq tF(v) \\
F(u) - (1-t)F(u) &\leq tF(v) \\
F(u) &\leq (1-t)F(u) + tF(v)
\end{aligned}$$

We are missing something above... Something that convexity argument comes down to...

$$\begin{aligned}
F(u) &\leq F((1-t)u + tv) \\
\lim_{t\to 0} \frac{F(u+t(v-u)) - F(u)}{t} &\geq 0
\end{aligned}$$

Problem 4.5

Let (x_n) be a weakly convergent sequence in a Hilbert space $\mathcal{H}$. What is a necessary and sufficient condition for (x_n) to be strongly convergent?

For strong convergence, we want

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Writing this out as an inner product, we get

$$\langle x_n - x, x_n - x\rangle = \langle x, x\rangle - 2\text{Re}\,\langle x_n, x\rangle + \langle x, x\rangle$$

Basically, we want

$$\|x_n\|^2 \to \|x\|^2$$

Problem 4.6

Suppose $a : H \times H \to \mathbb{R}$ is a continuous bilinear form. Suppose a is positive definite: $a(u,u) \geq 0 \ \forall\ u \in H$. Prove that $v \mapsto F(v) = a(v,v)$ is convex, C^1, and compute its derivative.

Hints:

1. Define $A \in \mathcal{L}(\mathcal{H})$ such that $a(u,v) = (Au, v) \ \forall\ u, v \in \mathcal{H}$
2. $t(1-t)a(u-v, u-v) \geq 0$

Problem 4.7

For a bounded self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$, prove that

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax \rangle|.$$

See page 198 in H&N.

Problem 4.8

Let (x_n) be a weakly convergent sequence in $\mathcal{H}$. Define P_{x_n} as the projection $\langle x_n, \cdot \rangle \frac{x_n}{\|x_n\|}$. What can we say about

(a) strong convergence of (P_{x_n})?

(b) norm convergence of (P_{x_n})?

(a) $x_n \rightharpoonup x$

$$\begin{aligned} P_{x_n} y &= \langle x_n, y \rangle \frac{x_n}{\|x_n\|} \\ \|P_{x_n} y\| &= |\langle x_n, y \rangle| \\ &\leq C(y) \quad \text{bounded} \end{aligned}$$

So there exists a subsequence $P_{x_n} \rightharpoonup \tilde{y} \in \mathcal{H}$. This means that $\langle P_{x_n} y, z \rangle \to \langle \tilde{y}, z \rangle \ \forall \ z \in \mathcal{H}$. Thus,

$$\langle x_n, y \rangle \frac{\langle x_n, z \rangle}{\|x_n\|} \to \langle \tilde{y}, z \rangle$$

Choose a subsequence of (x_n) such that $\|x_n\| \to M$. Then

$$\begin{aligned} \frac{1}{\|x_n\|} \langle x_n, y \rangle \langle x_n, z \rangle &\to \frac{1}{M} \langle x, y \rangle \langle x, z \rangle \\ \langle x, y \rangle \langle x, z \rangle \frac{1}{M} &= \langle \tilde{y}, z \rangle \qquad \forall \ z \in \mathcal{H} \\ \left\langle \frac{\langle x, y \rangle x}{M}, z \right\rangle &= \langle \tilde{y}, z \rangle \\ \tilde{y} &= \langle x, y \rangle \frac{x}{M} \\ P_{x_n} y \rightharpoonup \tilde{y} &= \langle x, y \rangle \frac{x}{M} \end{aligned}$$

$$\begin{aligned} \|P_{x_n} y - \tilde{y}\|^2 &= \langle P_{x_n} y - \tilde{y}, P_{x_n} y - \tilde{y} \rangle \\ &= \end{aligned}$$

From Joe:
Is there an operator $P_x : \mathcal{H} \to \mathcal{H}$ such that for each $y \in \mathcal{H}$

$$\lim_{n \to \infty} \|P_x(y) - P_{x_n}(y)\| = 0$$

This would imply that $P_{x_n}(y) \rightharpoonup P_x(y)$. Fix y. Consider

$$\langle P_{x_n}(y), z \rangle = \frac{\langle x_n, y \rangle \langle x_n, z \rangle}{\|x_n\|}$$

If $x_n \rightharpoonup x$ then we know that $\|x_n\| < M$ for some M.

From Chuan:
Let $\mathcal{H} = L^2(0,1)$, $x_n = 1 + \sin(n\pi x)$, $x_n \rightharpoonup 1$. Then

$$\begin{aligned} P_{x_n} f &= \langle x_n, f \rangle \frac{x_n}{\|x_n\|} \\ &= \int_0^1 (1 + \sin(n\pi x)) f(x)\, dx \\ &= \int_0^1 f(x)\, dx \frac{1 + \sin(n\pi x)}{\sqrt{\frac{3}{2}}} \end{aligned}$$

Apparently this is not a Cauchy sequence in L^2, so it does not converge strongly.

(b) Consider $x_n = e_n$ in $\ell^2(\mathbb{N})$. Then $\|P_{e_n} - P_{e_m}\| = \sqrt{2}$, so it is not a Cauchy sequence and hence not convergent.

Problem 4.9

Let $\mathcal{H} = L^2(\mathbb{T})$ and $A : \mathcal{H} \to \mathcal{H}$. Define

$$A(f) = \int_0^{2\pi} [\cos(x)\cos(y) + 2\cos(2x)\cos(y) + \cos(x)\cos(2y)]\, f(y)\, dy$$

What is the condition on $g \in \mathcal{H}$ for $Af = g$ to have a solution?

This might be the same as

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

If the range is closed then either there exists a solution or g is not $\perp$ to ker A^*.
This operator is compact.

Problem 4.10

Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{T})$ and

$$P = \frac{1}{2}(I + i\mathcal{F})$$
$$Q = \frac{1}{2}(I - i\mathcal{F})$$

Instead of the fourier transform, use a bounded linear transformation satisfying

$$F^2 = -I$$
$$F^* = -F$$

Prove that P and Q are orthogonal projections.

$P^2 \stackrel{?}{=} P$

$$\begin{aligned} P^2(x) &= \frac{1}{4}(I + iF)(I + iF) \\ &= \frac{1}{4}(I + 2IF - F^2) \\ &= \frac{1}{2}(I + iF) \end{aligned}$$

$P^* \stackrel{?}{=} P$

$$\begin{aligned} P^* &= (I + iF)^* \\ &= (I^* - iF^*) \\ &= (I + iF) \\ &= P \end{aligned}$$

Problem 4.11

$A : \mathcal{H} \to \mathcal{H}$, A is bounded and linear, and there exists $c > 0$ such that

$$c\|x\| \leq \|Ax\| \ \forall \ x$$

What can you say about ran A?

9-16-11

Problem 5.1

Let $\mathcal{H}$ be a separable Hilbert space. Suppose we have 2 orthonormal systems (bases), (e_n) and (f_n). Let (λ_n) be a bounded sequence of complex numbers. We have

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n.$$

1. Prove that T is a bounded linear operator, with $\|T\|_{\text{op}} = \sup |\lambda_n|$
2. T is compact iff $\lambda_n \to 0$ as $n \to \infty$.
3. If K is a compact operator on $\mathcal{H}$ then there exists orthonormal (e_n) and (f_n) and a sequence of complex numbers (λ_n) converging to zero such that

$$Kx = \sum \lambda_n (x_n, e_n) f_n \qquad \forall\, x \in \mathcal{H}$$

1.

$$\begin{aligned} \|Tx\|^2 &\leq \left\| \sum \lambda_n (x, e_n) f_n \right\|^2 \\ &\leq \sum |\lambda_n|^2 (x, e_n)^2 \\ &\leq (\sup |\lambda_n|)^2 \|x\|^2 \end{aligned}$$

Let λ_{n_j} be a subsequence such that $|\lambda_{n_j}| \to A = \sup |\lambda_n|$. Then

$$\|Te_{n_j}\|^2 = \|\lambda_{n_j}\|^2 \to A^2$$

2. If T is compact, then it takes weakly convergent sequences to strongly convergent sequences. $(e_n) \rightharpoonup 0$, so $\|Te_n\| \to 0$. This is the case because $\|Te_n\| = |\lambda_n| \to 0$.

Let

$$\begin{aligned} T_N &= \sum_{n=1}^{N} \lambda_n (x, e_n) f_n \\ \|T_N - T\|_{\text{op}} &= \left\| \sum_{N+1}^{\infty} \lambda_n (x, e_n) f_n \right\|_{\text{op}} \\ &= \sup_{n>N} |\lambda_n| \\ &\to 0 \end{aligned}$$

3. Apply the spectral decomposition theorem to K^*K, which is positive. ($\langle x, K^*Kx \rangle = \langle Kx, Kx \rangle \geq 0$)

Problem 5.2

"Extremely important problem!"

$L^2(0,1)$, complex-valued.

$$\langle f, g\rangle_{L^2} = \int_0^1 f(x)\overline{g(x)}\, dx$$

Define $T : L^2(0,1) \to L^2(0,1)$ by

$$Tf(x) = \int_0^x f(t)\, dt, \qquad x \in [0,1].$$

1. Show that T is bounded and compact.
2. Show that T has no eigenvalues. This means that if $Tf = \lambda f,\ \lambda \in \mathbb{C},\ f \in L^2(0,1)$, then $f = 0$.
3. Find $\lim_{n\to\infty} \|T^n\|$ and using this, prove that the spectrum of T is $\{0\}$. i.e. $T - \lambda I$ is an isomorphism of $L^2(0,1)$ onto itself iff $\lambda \neq 0$.

1. Bounded:

$$\begin{aligned}|Tf(x)| &\le \int_0^x |f(t)|\, dt \\ &\le \int_0^1 |f(t)|\, dt = \|f\|_{L^1} \\ &\le \|f\|_{L^2}\|1\|_{L^2} = \|f\|_{L^2} \\ \|T\| &\le 1.\end{aligned}$$

Compact:
T is compact if for every bounded subset $\{f_n\} \subset L^2(0,1)$, $\|f_n\|_{L^2} \le M$, it is true that $\{Tf_n\}$ is precompact. By Arzela-Ascoli, $\{Tf_n\}$ is precompact if it is bounded and equicontinuous. Bounded is easy:

$$\|Tf_n\|_{L^2} \le \|T\|_{\text{op}}\|f_n\| \le M.$$

To show equicontinuous, fix x. Then

$$\begin{aligned}|Tf_n(x+\delta) - Tf_n(x)| &= \left|\int_0^{x+\delta} f_n(t)\, dt - \int_0^x f_n(t)\, dt\right| \\ &= \left|\int_x^{x+\delta} f_n(t)\, dt\right| \\ &\le \int_0^1 \mathbf{1}_{[x,x+\delta]}|f_n(t)|\, dt \\ &\le \left(\int_0^1 \mathbf{1}_{[x,x+\delta]}\, dt\right)^{1/2} \|f_n\|_{L^2} \\ &\le \sqrt{\delta}M.\end{aligned}$$

Thus, $\{f_n\}$ is equicontinous and bounded, so $\{Tf_n\}$ is precompact, so T is a compact operator.

2. Set $Tf = \lambda f$ and differentiate with respect to t:

$$\begin{aligned}
\frac{d}{dt}\int_0^x f(t)\,dt &= \frac{d}{dt}\lambda f(x) \\
f(x) - f(0) &= 0 \\
f(x) &= f(0) \\
\int_0^0 f(t)\,dt &= 0 = \lambda f(0) \\
f(0) &= 0 = f(x)
\end{aligned}$$

3. Note that $\left|\int\limits_0^x f(t)\,dt\right| \leq \|f\|_{L^2}$. We compute:

$$\begin{aligned}
|T^2 f(x)| &= \left|\int_0^x \int_0^{x_1} f(t)\,dt\,dx_1\right| \\
&\leq \int_0^x \|f\|_{L^2}\,dx_1 \\
&= x\|f\|_{L^2} \\
|T^3 f(x)| &= \left|\int_0^x \int_0^{x_2} \int_0^{x_1} f(t)\,dt\,dx_1\,dx_2\right| \\
&\leq \int_0^x \int_0^{x_2} \|f\|_{L^2}\,dx_1\,dx_2 \\
&= \int_0^x x_2\|f\|_{L^2}\,dx_2 \\
&= \frac{x^2}{2}\|f\|_{L^2}
\end{aligned}$$

From here I claim that

$$\|T^n f\| \leq \frac{1}{(n-1)!}\|f\|_{L^2}.$$

Thus, $\lim\limits_{n\to\infty} \|T^n\| = 0 = r(T)$ (the spectral radius). Thus, the only element in the spectrum of T is 0.

Problem 5.3

Let C be a nonempty closed and convex bounded subset of a Banach space X. Let $f : C \to C$ satisfy

$$\|f(x) - f(y)\| \leq \|x - y\| \qquad \forall\ x, y \in C.$$

Then there exists in C an approximate fixed point sequence of f. Assume $0 \in C$.

An *approximate fixed point sequence* is a sequence (x_n) such that for every $S \subset X$, $f : S \to S$ and

$$\lim_{n\to\infty} \|x_n - f(x_n)\| = 0.$$

Hint: for $\epsilon > 0$, let $C_\epsilon = \{(1-\epsilon)x \mid x \in C\}$, $C_\epsilon \subset C$.

Let

$$f_\epsilon = (1-\epsilon)f.$$

Then f_e is a contraction mapping because

$$\begin{aligned}\|f_\epsilon(x) - f_\epsilon(y)\| &= \|(1-\epsilon)f(x) - (1-\epsilon)f(y)\| \\ &= (1-\epsilon)\|f(x) - f(y)\| \\ &\leq (1-\epsilon)\|x - y\|.\end{aligned}$$

Therefore, there exists $x_\epsilon \in C_\epsilon$ such that $f_\epsilon(x_\epsilon) = x_\epsilon$.

$$\begin{aligned}\|x_\epsilon - f(x_\epsilon)\| &= \left\| x_\epsilon - \frac{x_\epsilon}{1-\epsilon} \right\| \\ &= \|x_\epsilon\| \left| 1 - \frac{1}{1-\epsilon} \right|\end{aligned}$$

Problem 5.4

True or False. Let A be a closed, bounded, and convex subset of $C[0,1]$.

$$A = \{x \in C[0,1] \mid 0 = x(0) \leq x(t) \leq x(1) = 1\}$$

The mapping $T : A \to A$ defined by

$$T(x)(t) = tx(t)$$

has a fixed point.

Assume we have a fixed point: $Tf(t) = f(t)$. Then $tf(t) = f(t)$, and so $0 = f(t)(1-t)$. Thus, $f(t) = 0$ on $[0,1]$. But this does not satisfy the criteria. $\Rightarrow\Leftarrow$

Problem 5.5

Let $1 < p < \infty$, $q = \frac{p}{p-1}$. Suppose that we have $f_n \rightharpoonup f$ in $L^p(\Omega)$, $g_n \to g$ in $L^q(\Omega)$. Show

$$f_n g_n \rightharpoonup fg \qquad \text{in } \mathcal{D}'(\Omega) \text{ (i.e. distributionally)}$$

This is the same as saying $f_n g_n \overset{*}{\rightharpoonup} fg$ in $L^\infty(\Omega)$.

Add and subtract something, use Hölder's inequality.

$$f_n g_n - fg = f_n g_n - f g_n + f g_n - fg$$

Here goes nothing... For $\phi \in L^1(\Omega)$, we want to show that the following converges to zero as $n \to \infty$.

$$\begin{aligned}
\int_\Omega f_n g_n \phi \, dx - \int_\Omega f g \phi \, dx &= \int_\Omega [f_n g_n - fg] \phi \, dx \\
&= \int_\Omega [(f_n g_n - f g_n) + (f g_n - fg)] \phi \, dx \\
&= \int_\Omega (f_n g_n - f g_n) \phi \, dx + \int_\Omega (f g_n - fg) \phi \, dx \\
&= \int_\Omega g_n (f_n - f) \phi \, dx + \int_\Omega f (g_n - g) \phi \, dx
\end{aligned}$$

Problem 5.6

Consider a subspace of $\ell^\infty(\mathbb{Z})$. $c_0 \subset \ell^\infty$ are the bilateral sequences, $x = (x_n)$, such that $x_n \to 0$ as $|n| \to \infty$. $\mathcal{F} : L^1(S^1) \to c_0$, meaning that the Fourier coefficients of a function f is in c_0. $\mathcal{F}$ is bounded and injective. Prove that $\mathcal{F}(L^1(S^1)) \neq c_0$.

Hint: prove by contradiction. Use the open mapping theorem. Construct a sequence

$$D_N = \sum_{-N}^{N} e^{inx} = \text{Dirichlet kernel}$$

Try to integrate this and find that it blows up.

Problem 5.7

Let $T \in \mathcal{L}(X, Y)$ be compact and $x_n \rightharpoonup x$. Show $T(x_n) \to Tx$ in Y. Hint: show $T(x_n) \rightharpoonup T(x)$ in Y.

Let $f \in Y'$. Define

$$g(x) = f(T(x)).$$

Then

$$\|g\| \leq \|f\| \|T\|$$

Thus, $g \in X'$.

$$\begin{aligned} g(x_n) - g(x) &\to 0 \\ g(x_n - x) &\to 0 \\ f(T(x_n - x)) &\to 0 \end{aligned}$$

So $Tx_n \rightharpoonup Tx$. Now assume that $Tx_n \not\to Tx$. Then there exists a subsequence (x_{n_k}) such that for every $\epsilon > 0$,

$$\|Tx_{n_k} - Tx\| \geq \epsilon.$$

(x_{n_k}) is a bounded sequence. T is compact, so $Tx_{n_{k_l}} \to y$ in Y. This gives us a contradiction.

Problem 5.8

We have separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. $A : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded and linear. Suppose we have a linear operator $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and compact operators E_i on $H_i,\ i = 1, 2$ such that

$$BA = I_{\mathcal{H}_1} - E_1$$
$$AB = I_{\mathcal{H}_2} - E_2.$$

Prove that $N(A)$ is finite dimensional, and that $R(A)$ is closed in $\mathcal{H}_2$.

Problem 5.9

$\mathcal{H}$ is a separable Hilbert space. (e_n), (f_n) are orthonormal bases. The closure of the span of $\{f_n\} = \mathcal{H}$.

1. Prove that if $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$ then $\{e_n\}$ is also complete and orthonormal.

2. Suppose $\sum_{n=1}^{\infty} < \infty$. Prove that $\{e_n\}$ is a complete orthonormal basis.

Problem 5.10

Suppose $f \in L^1(0,1)$ but $f \notin L^2(0,1)$. Find a complete orthonormal basis $\{\phi_n\}$ for $L^2(0,1)$ such that each $\phi_n \in C^0([0,1])$ and such that

$$\int_0^1 f(x)\phi_n(x)\,dx = 0 \qquad \forall\ n$$

Problem 5.11

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces. Suppose that $A : \mathcal{H}_1 \to \mathcal{H}_2$ and it is continuous, linear, and injective. Let (v_j) be a bounded sequence in $\mathcal{H}_1$ such that (Av_j) converges strongly in $\mathcal{H}_2$ to some w. Prove that there exists $v \in \mathcal{H}_1$ such that $v_j \rightharpoonup v$ in $\mathcal{H}_1$.

Problem 5.12

Let $\mathcal{H} = \bigoplus_{j=1}^{\infty} \mathcal{H}_j$, $\mathcal{H}_j$ is finite-dimensional. That is, for $v \in \mathcal{H}$, $v = \sum_{j=1}^{\infty} v_j$, $v_j \in \mathcal{H}_j$. Let $C = (c_1, c_2, c_3, \ldots)$, $c_j > 0$.

$$A_{C,\mathcal{H}} = \left\{ v \;\middle|\; \|v_j\| \le c_j \right\} \subset \mathcal{H}$$

1. Prove $C \in \ell^2$ iff $A_{C,\mathcal{H}}$ is compact in $\mathcal{H}$.
2. Prove that every compact subset $K \subset \mathcal{H}$ is contained in *some* $A_{C,\mathcal{H}}$ for some $\mathcal{H}$ and $C \in \ell^2$.

Problem 5.13

Let G be an unbounded set in $(0, \infty)$. Define

$$D = \left\{ x \in (0, \infty) \;\middle|\; nx \in G \text{ for infinitely many } n \right\}$$

Prove that D is dense in $(0, \infty)$. (n is not restricted to the natural numbers)