Solving a Linear System

- $\tau = \text{trace}(A) = a + d = \lambda_1 + \lambda_2$
- $\Delta = \det(A) = ad - bc = \lambda_1\lambda_2$
- $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$

Classification of Fixed Points

1. $\Delta < 0$: the eigenvalues are *real* and have opposite signs; the fixed point is a *saddle point*
2. $\Delta > 0$:
   (a) The eigenvalues are *real* with the same sign (*stable/unstable nodes*)
      i. $\lambda_1, \lambda_2 > 0 \ \Rightarrow$ unstable
      ii. $\lambda_1, \lambda_2 < 0 \ \Rightarrow$ stable
   (b) The eigenvalues are *complex*: $\lambda_{1,2} = \alpha \pm i\omega$
      i. $\alpha > 0 \ \Rightarrow$ *growing (unstable) spirals*
      ii. $\alpha = 0 \ \Rightarrow$ *circles*
      iii. $\alpha < 0 \ \Rightarrow$ *decreasing (stable) spirals*
      Note: direction (clockwise/counterclockwise) depends on initial conditions and must be checked.
   (c) $\tau^2 = 4\Delta \ \Rightarrow$ 1 eigenvalue
3. $\Delta = 0$: at least one of the eigenvalues is zero
   - The origin is not an isolated fixed point. There is either a whole line of fixed points, or a whole plane of fixed points if $A = 0$.

Polar Coordinates Identity: $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$

Index Theory

Define $\phi = \tan^{-1}(\dot{y}/\dot{x})$. The *index* of a closed curve C is defined as $I_C = \frac{1}{2\pi}[\phi]_C$, where $[\phi]_C$ is the net change in ϕ over one circuit, i.e. the number of counterclockwise revolutions made by the vector field as $\mathbf{x}$ moves once counterclockwise around C.

Properties of the Index

1. Suppose that C can be continuously deformed into C' without passing through a fixed point. Then $I_C = I_{C'}$.
2. If C doesn't enclose any fixed points, then $I_C = 0$.
3. If we reverse all the arrows in the vector field by changing $t \to -t$, the index is unchanged.
4. Suppose that the closed curve C is actually a *trajectory* for the system, i.e. C is a closed orbit. Then $I_C = +1$.

More Index Info

- The index of a fixed point, $\mathbf{x}^*$, is defined as I_C, where C is any closed curve that encloses $\mathbf{x}^*$ and no other fixed points.
  - $I = +1$ for a stable node, unstable node, spiral, center, degenerate node, and star
  - $I = -1$ for a saddle point
- **Theorem:** If a closed curve C surrounds n isolated fixed points, then $I_C = I_1 + I_2 + \ldots + I_n$.
- **Theorem:** Any closed orbit in the phase plane must enclose fixed points whose indices sum to $+1$.

- Closed orbits are impossible for the "rabbit vs. sheep" system.

Limit Cycles

A *limit cycle* is an isolated closed trajectory. *Isolated* means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle. Limit cycles can be stable, unstable, or half-stable.

Ruling Out Closed Orbits

1. **Theorem:** Closed orbits are impossible in gradient systems ($\dot{\mathbf{x}} = -\nabla V$).

2. A *Liapunov function* is a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

   (a) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$. (We say that V is *positive definite.*)
   (b) $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow "downhill" toward $\mathbf{x}^*$.)

   The system has no closed orbits. (Sample Liapunov function: $V(x, y) = x^2 + ay^2$)

3. **Dulac's Criterion:** Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable, real-valued function $g(\mathbf{x})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign throughout R, then there are no closed orbits lying in R.

   - Candidates that occasionally work are $g = 1,\ 1/x^a y^b,\ e^{ax},$ and e^{ay}.

Poincaré-Bendixson Theorem

1. R is a closed, bounded subset of the plane

2. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R

3. R does not contain any fixed points

4. There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time

   - The standard trick for (4) is to construct a *trapping region* R, i.e. a closed connected set such that the vector field points inward everywhere on the boundary of R

If (1)-(4) are satisfied, then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

Liénard Systems

Liénard's equation has the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$.
Liénard's Theorem: Suppose that $f(x)$ and $g(x)$ satisfy the following conditions:

1. $f(x)$ and $g(x)$ are continuously differentiable for all x

2. $g(-x) = -g(x)$ (g is odd)

3. $g(x) > 0$ for $x > 0$

4. $f(-x) = f(x)$ (x is even)

5. The odd function $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$

Then the system has a unique, stable limit cycle surrounding the origin in the phase plane.

Two-Timing

In weakly nonlinear oscillators, introduce multiple time scales: $\tau = t,\ T = \epsilon t$. Then

$$\begin{aligned} x(t,\epsilon) &= x_0(\tau,T) + \epsilon x_1(\tau,T) + O(\epsilon^2)\ x_0(\tau,T) + O(\epsilon) \\ \dot{x} &= \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T} = \partial_\tau x + \epsilon \partial_T x \\ \ddot{x} &= \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + O(\epsilon^2) \end{aligned}$$

Averaged Equations: $\ddot{x} + x + \epsilon h(x,\dot{x}) = 0$

$$\begin{aligned} & O(1): \ \partial_{\tau\tau} x_0 + x_0 = 0, \qquad O(\epsilon): \ \partial_{\tau\tau} x_1 + x_1 = -\partial_{\tau T} x_0 - h \\ & x_0 = r(T)\cos(\tau + \phi(T)) = r(T)\cos[(1+\epsilon\phi')t + \theta] \\ & r' = \langle h \sin\theta \rangle, \qquad r\phi' = \langle h\cos\theta\rangle \\ & \omega = \frac{d}{dt}(\theta + \phi) = 1 + \epsilon\phi' \end{aligned}$$

$\langle \cos \rangle = \langle \sin \rangle = 0$, $\langle \sin\ \cos \rangle = 0$, $\left\langle \cos^3 \right\rangle = \left\langle \sin^3 \right\rangle = 0$, $\left\langle \cos^{2n+1} \right\rangle = \left\langle \sin^{2n+1} \right\rangle = 0$, $\left\langle \cos^2 \right\rangle = \left\langle \sin^2 \right\rangle = \frac{1}{2}$, $\left\langle \cos^4 \right\rangle = \left\langle \sin^4 \right\rangle = \frac{3}{8}$, $\left\langle \cos^2\ \sin^2 \right\rangle = \frac{1}{8}$, $\left\langle \cos^{2n} \right\rangle = \left\langle \sin^{2n} \right\rangle = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}$, $n \geq 1$, $\left\langle \cos^2\ \sin^4 \right\rangle = \frac{1}{16}$, $\left\langle \cos^3\ \sin \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \cos^3\theta \sin\theta\, d\theta = 0$

2-D Bifurcations

- Saddle-Node: $\dot{x} = \mu - x^2,\ \dot{y} = -y$
- Transcritical: $\dot{x} = \mu x - x^2,\ \dot{y} = -y$
- Pitchfork (Supercritical): $\dot{x} = \mu x - x^3,\ \dot{y} = -y$
- Pitchfork (Subcritical): $\dot{x} = \mu x + x^3,\ \dot{y} = -y$

1-D Bifurcations

- Saddle-Node: $\dot{x} = r \pm x^2$
- Transcritical: $\dot{x} = rx - x^2$
- Pitchfork (Supercritical): $\dot{x} = rx - x^3$
- Pitchfork (Subcritical): $\dot{x} = rx + x^3$

Example 6.8.5: Index Example 1

Show that closed orbits are impossible for the "rabbits vs. sheep" system

$$\dot{x} = x(3 - x - 2y), \qquad \dot{y} = y(2 - x - y)$$

Solution: The system has 4 fixed points: (0,0) = unstable node; (0,2) and (3,0) stable nodes; and (1,1) saddle point. If C is a closed orbit, then $I_C = 1$. The stable nodes have index 1, but cannot be closed orbits because a closed orbit would intersect the x or y axes, which are themselves trajectories, and trajectories cannot intersect.

Example 6.8.6: Index Example 2

Show that the system $\dot{x} = xe^{-x},\ \dot{y} = 1 + x + y^2$ has no closed orbits.

Solution: This system has no fixed points: if $\dot{x} = 0$, then $x = 0$ and so $\dot{y} = 1 + y^2 \neq 0$. Then closed orbits cannot exist because **Theorem:** any closed orbit in the phase plane must enclose fixed points whose indices sum to +1.

Example 7.1.1: A simple limit cycle

Consider the system

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1$$

where $r \geq 0$. We see that $r^* = 0$ is an unstable fixed point and $r^* = 1$ is stable. Hence, all trajectories (except $r^* = 0$) approach the unit circle monotonically.

Example 7.1.2: van der Pol Oscillator

The van der Pol equation is

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

where $\mu \geq 0$ is a parameter. The van der Pol equation has a unique, stable limit cycle for each $\mu > 0$.

Example 7.2.1

Show that there are no closed orbits for the system $\dot{x} = \sin y,\ \dot{y} = x \cos y$.

Solution: The system is a gradient system with potential function $V(x, y) = -x \sin y$, and so it has no closed orbits.

Example 7.2.2

Show that the damped oscillator $\ddot{x} + (\dot{x})^3 + x = 0$ has no periodic solutions.

Solution: Suppose that there were a periodic solution $x(t)$ of period T. Consider the energy function $E(x, \dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$. After one cycle, x and $\dot{x}$ return to their starting values, and therefore $\Delta E = 0$ around any closed orbit. On the other hand, $\Delta E = \int_0^T E\,dt$. Note that $\dot{E} = \dot{x}(x + \ddot{x}) = -\dot{x}^4 \leq 0$. Therefore $\Delta E \leq 0$ with equality if $\dot{x} \equiv 0$, but this would mean the trajectory is a fixed point. Hence, there are no periodic solutions.

Example 7.2.3

By constructing a Liapunov function, show that the system $\dot{x} = -x + 4y,\ \dot{y} = -x - y^3$ has no closed orbits.

Solution: Consider $V(x, y) = x^2 + ay^2$. Then $\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x(-x + 4y) + 2ay(-x - y^3) = -2x^2 + (8 - 2a)xy - 2ay^4$. If we choose $a = 4$, the xy term disappears and $\dot{V} = -2x^2 - 8y^4$. By inspection, $V > 0$ and $\dot{V} < 0$ for all $(x, y) \neq (0, 0)$. Hence $V = x^2 + 4y^2$ is a Liapunov function and so there are no closed orbits.

Example 7.2.3

Show that the system $\dot{x} = x(2 - x - y),\ \dot{y} = y(4x - x^2 - 3)$ has no closed orbits in the positive quadrant.

Solution: Pick $g = 1/xy$. Then $\nabla \cdot (g\dot{\mathbf{x}}) < 0$. Since the positive quadrant is simply connected and g and $\mathbf{f}$ satisfy the smoothness conditions, Dulac's criterion implies that there are no closed orbits in the positive quadrant.

Example 7.3.1

Consider the system

$$\dot{r} = r(1 - r^2) + \mu r \cos\theta, \qquad \dot{\theta} = 1$$

When $\mu = 0$, there's a stable limit cycle at $r = 1$, as discussed in example 7.1.1. Show that a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small.

Solution: We seek 2 concentric circles with radii r_{min} and r_{max} such that $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle. Then the annulus $0 < r_{\text{min}} \leq r \leq r_{\text{max}}$ will be our desired trapping region. $r_{\text{min}} = 0.999\sqrt{1 - \mu}$ and $r_{\text{max}} = 1.001\sqrt{1 + \mu}$ will work. Therefore, a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1 - \mu} < r < 1.001\sqrt{1 + \mu}$.

Example 7.3.2

Construct a trapping region for

$$\dot{x} = -x + ay + x^2 y, \qquad \dot{y} = b - ay - x^2 y$$

Solution: Find the nullclines:

$$\dot{x} = 0 \text{ on } y = x/(a + x^2), \qquad \dot{y} = 0 \text{ on } y = b/(a + x^2)$$

Then we can draw a pseudo-right-triangular trapping region in the first quadrant and show that the vector field points inward along the edge of this region.

Example 7.4.1

Show that the van der Pol equation $(\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0)$ has a unique, stable limit cycle.

Solution: The van der Pol equation has $f(x) = \mu(x^2 - 1)$ and $g(x) = x$, so conditions 1-4 of Liéndard's theorem are satisfied. To check condition 5, notice that

$$F(x) = \mu\left(\frac{1}{3}x^3 - x\right) = \frac{1}{3}\mu x(x^2 - 3)$$

Hence condition 5 is satisfied for $a = \sqrt{3}$. Thus, the van der Pol equation has a unique, stable limit cycle.

Example 7.5.1

Give a phase plane analysis of the van der Pol equation for $\mu \gg 1$.

Solution: It proves convenient to introduce different phase plane variables.0 Write

$$\dot{x} = w - \mu F(x), \qquad \dot{w} = -x \quad \Rightarrow \quad \dot{x} = \mu[y - F(x)], \qquad \dot{y} = -\frac{1}{\mu}x \quad \left(\text{where } y = \frac{w}{\mu}\right)$$

Example 7.6.1

Use two-timing to approximate the solution to the damped linear oscillator $\ddot{x} + 2\epsilon\dot{x} + x = 0$, with initial conditions $x(0) = 0,\ \dot{x}(0) = 1$.

Solution:

$$\partial_{\tau\tau}x_0 + \epsilon(\partial_{\tau\tau}x_1 + 2\partial_{T\tau}x_0) + 2\epsilon\partial_\tau x_0 + x_0 + \epsilon x_1 + O(\epsilon^2) = 0$$

$$\text{Collect } \epsilon^0 \text{ and } \epsilon^1 \text{ terms: } O(1): \ \partial_{\tau\tau}x_0 + x_0 = 0 \tag{1}$$

$$O(\epsilon): \ \partial_{\tau\tau}x_1 + 2\partial_{T\tau}x_0 + 2\partial_\tau x_0 + x_1 = 0 \tag{2}$$

The solution of (1) is $x_0 = A\sin\tau + B\cos\tau$, where $A = A(T),\ B = B(T)$. Substituting this into (2) yields

$$\partial_{\tau\tau}x_1 + x_1 = -2(A' + A)\cos\tau + 2(B' + B)\sin\tau$$

Set the resonant coefficients to zero to get

$$A' + A = 0 \quad \Rightarrow \quad A(T) = A(0)e^{-T}$$
$$B' + B = 0 \quad \Rightarrow \quad B(T) = B(0)e^{-T}$$

Use the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$ and the 2-timing formula for $\dot{x}$ to get that $A = 1$ and $B = 0$, and hence $x = e^{-\epsilon t}\sin t + O(\epsilon)$.

Example 7.6.2

Use 2-timing to show that the van der Pol oscillator, $\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0$, has a stable limit cycle that is nearly circular, with a radius $2 + O(\epsilon)$ and a frequency $\omega = 1 + O(\epsilon^2)$.

Solution: Using the 2-timing formula for $\dot{x}$ and collecting powers of ϵ we get

$$O(1): \ \partial_{\tau\tau}x_0 + x_0 = 0 \tag{3}$$

$$O(\epsilon): \ \partial_{\tau\tau}x_1 + x_1 = -2\partial_{\tau T}x_0 - (x_0^2 - 1)\partial_\tau x_0 \tag{4}$$

The solution to (3) is $x_0 = r(T)\cos(\tau + \phi(T))$. Plugging this into (4) yields

$$\begin{aligned}\partial_{\tau\tau}x_1 + x_1 &= 2(r'\sin(\tau+\phi) + r\phi'\cos(\tau+\phi)) - r\sin(\tau+\phi)[r^2\cos^2(\tau+\phi) - 1] \\ &= \left[-2r' + r - \frac{1}{4}r^3\right]\sin(\tau+\phi) + [-2r\phi']\cos(\tau+\phi) - \frac{1}{4}r^3\sin[3(\tau+\phi)]\end{aligned}$$

To avoid secular terms, we require

$$-2r' + r - \frac{1}{4}r^3 = 0 = -2r\phi'$$

First consider the LHS. This may be rewritten as a vector field $r' = \frac{1}{8}r(4 - r^2)$, and we can see that $r(T) \to 2$ as $T \to \infty$. The RHS implies $\phi' = 0$, so $\phi(T) = \phi_0$. Hence $x_0(\tau, T) \to 2\cos(\tau + \phi_0)$ and therefore $x(t) \to 2\cos(t + \phi_0) + O(\epsilon)$ as $t \to \infty$. To find the frequency, let $\theta = t + \phi(T)$. Then

$$\omega = \frac{d\theta}{dt} = 1 + \frac{d\theta}{dT}\frac{dT}{dt} = 1 + \epsilon\phi' = 1$$

Hence, $\omega = 1 + O(\epsilon^2)$.

Example 7.6.3

Consider the van der Pol equation $\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0$, subject to the initial conditions $x(0) = 1,\ \dot{x}(0) = 0$.

Find the averaged equations, and then solve them to obtain an approximate formula for $x(t,\epsilon)$.
Solution: The van der Pol equation has $h = (x^2 - 1)\dot{x} = (r^2\cos^2\theta - 1)(-r\sin\theta)$. Hence

$$r' = \langle h\sin\theta\rangle = \frac{1}{2}r - \frac{1}{8}r^3$$
$$r\phi' = \langle h\cos\theta\rangle = 0$$

The initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ imply $r(0) = \sqrt{x(0)^2 + \dot{x}(0)^2} = 1$ and $\phi(0) \approx \tan^{-1}(\dot{x}(0)/x(0)) - \tau = 0$. Since $\phi' = 0$, we find $\phi(T) \equiv 0$. To find $r(T)$, we solve $r' = \frac{1}{2}r - \frac{1}{8}r^3$ subject to $r(0) = 1$. The differential equation separates to

$$\int \frac{8dr}{r(4-r^2)} = \int dT$$
$$r(T) = 2(1+3e^{-T})^{-1/2}$$

Hence $x(t,\epsilon)\ x_0(\tau,T) + O(\epsilon) = 2(1+3e^{-T})^{-1/2}\cos t + O(\epsilon)$.

Example 7.6.4

Find an approximate relation between the amplitued and frequency of the Duffing oscillator $\ddot{x} + x + \epsilon x^3 = 0$, where ϵ can have either sign.
Solution: Here $h = x^3 = r^3\cos\theta$, and $r' = \langle h\sin\theta\rangle = r^3\left\langle\cos^3\theta\sin\theta\right\rangle = 0$ and $r\phi' = \langle h\cos\theta\rangle = r^3\left\langle cos^4\theta\right\rangle = \frac{3}{8}r^3$. Hence $r(T) \equiv a$ for some constant a, and $\phi' = \frac{3}{8}a^2$. As in example 7.6.2, the frequency is given by $\omega = 1 + \epsilon\phi' = 1 + \frac{3}{8}\epsilon a^2 + O(\epsilon^2)$.

Example 8.1.1

$$\dot{x} = -ax + y, \qquad \dot{y} = \frac{x^2}{1+x^2} - by$$

Show that this system has 3 fixed points when $a < a_c$ and show that two of these fixed points coalesce in a saddle-node bifurcation when $a = a_c$.
Solution: The nullclines are given by

$$y = ax, \qquad y = \frac{x^2}{b(1+x^2)} \qquad \Rightarrow \quad \text{they intersect at} \quad ax = \frac{x^2}{b(1+x^2)}$$

One solution is $x^* = y^* = 0$. Other intersections are given by

$$x^* = \frac{1 \pm \sqrt{1 - 4a^2b^2}}{2ab}$$

if $1 - 4a^2b^2 > 0$. These solutions coalesce when $2ab = 1$, so $a_c = 1/2b$. The Jacobian matrix has trace $\tau = -(a+b) < 0$, so all fixed points are either sinks or saddles, depending on the determinant Δ. $(0,0)$ is a stable node. When we have 2 fixed points, the larger one is a stable node and the smaller one is a saddle point.

Example 8.1.2

Plot the phase portraits for the supercritical pitchfork system $\dot{x} = \mu x - x^3,\ \dot{y} = -y$, for $\mu < 0,\ \mu = 0,\ \mu > 0$.
Solution: For $\mu < 0$ the only fixed point is a stable node at the origin. For $\mu = 0$ the origin is still stable, but now we have very slow decay along the x-direction instead of exponential decay. For $\mu > 0$ the origin loses stability and gives birth to two new stable fixed points symmetrically located at $(x^*, y^*) = (\pm\sqrt{\mu}, 0)$. By computing the Jacobian at each point, you can check that the origin is a saddle and the other two fixed points are stable nodes.

Example 8.1.3

Show that a supercritical pitchfork bifurcation occurs at the origin in the system $\dot{x} = \mu x + y + \sin x,\ \dot{y} = x - y$ and determine the bifucation value μ.
Solution: The origin is a fixed point for al μ, and its Jacobian has $\tau = \mu$ and $\Delta = -(\mu + 2)$. The origin is is a stable fixed point if $\mu < -2$ and a saddle if $\mu > -2$, so a pitchfork bifurcation occurs at $\mu = -2$. The

fixed points satisfy $y = x$ and hence $(\mu+1)x + \sin x = 0$. Suppose x is small and nonzero and expand this as a power series to get:

$$(\mu+1)x + x - \frac{1}{3!}x^3 + O(x^5) = 0$$

Hence there is a pair of fixed points with $x^* = \pm\sqrt{6(\mu+2)}$ for μ slightly greater than -2, and hence a supercritical pitchfork bifurcation occurs at -2. (If the bifurcation had been subcritical, the pair of fixed points would exist when the origin was stable, not after it has become a saddle.) Because the bifurcation is supercritical, we know the new fixed points are stable without having to check.

Example 3/11/11

$$\dot{x} = \mu - x^2 \qquad \dot{y} = -y$$

$$\begin{aligned} \mu > 0 &\Rightarrow \mathbf{x}^* = (\pm\sqrt{\mu}, 0) \\ \mu = 0 &\Rightarrow \mathbf{x}^* = (0,0) \\ \mu < 0 &\Rightarrow \text{bottleneck, no fixed points} \end{aligned}$$

Problem 7.2.12

Show that $\dot{x} = -x + 2y^3 - 2y^4,\ \dot{y} = -x - y + xy$ has no periodic solutions.
Solution: Set $V = x^m + ay^n$. Then $\dot{V} = -mx^m + 2mx^{m-1}y^3 - 2mx^{m-1}y^4 - any^{n-1}x - any^n + anxy^n$. Setting m, n even and $a = 1$ gives us $\dot{V} = -2x^2 - 4y^4 \leq 0$. Thus, V is a Liapunov function and so the system has no closed orbits.

Problem 7.3.1

Using the Poincaré-Bendixson Theorem, show that $\dot{x} = x - y - x(x^2 + 5y^2),\ \dot{y} = x + y - y(x^2 + y^2)$ has a limit cycle somewhere in the trapping region $r_1 \leq r \leq r_2$.
Solution: Analysis of the Jacobian shows that the origin is an unstable spiral. Converting to polar coordinates, we get $\dot{r} = r - r^3 - 4r^3\cos^2\theta\sin^2\theta$ and $\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta$. Our annulus is the region $\frac{1}{\sqrt{2}} < r < 1$, and by the Poincaré-Bendixson Theorem the system has a limit cycle in this region.

Problem 7.3.3

Show that the system $\dot{x} = x - y - x^3,\ \dot{x} + y - y^3$ has a periodic solution.
Solution: Converting to polar coordinates gives us $\dot{r} = r - r^3(\cos^4\theta + \sin^4\theta)$ and $\dot{\theta} = 1 + \frac{1}{r^2}\sin(2\theta)\cos(2\theta)$. Note that $\frac{1}{2} \leq \cos^4\theta + \sin^4\theta \leq 1$. Our annulus is the region $1 < r < \sqrt{2}$.

Problem 7.3.5

Show that the system $\dot{x} = -x - y + x(x^2 + 2y^2),\ \dot{y} = x - y + y(x^2 + 2y^2)$ has at least one periodic solution.
Solution: Converting to polar coordinates gives us $\dot{r} = r^3 + r^3\sin^2\theta - r$ and $\dot{\theta} = 1$. When we solve for our annulus we get $1 < r < \frac{1}{\sqrt{2}}$, which is bogus. However, the system is invariant in time, and by running time in reverse we make the vectors point in the opposite direction. Therefore we can still use the Poincaré-Bendixson Theorem.

Sample Problem 3a

Use two-timing to solve $\ddot{x} + 2\epsilon\dot{x} + x = 0$, with the initial conditions $x(0) = 0,\ \dot{x}(0) = 1$.
Solution: $h = 2\dot{x} = -2r\sin\theta$. $\partial_T r = r' = \langle h\sin\theta\rangle = -2r\left\langle\sin^2\theta\right\rangle = -r$. $r\partial_T r = r\phi' = \langle h\cos\theta\rangle = -2r\langle\sin\theta\cos\theta\rangle = 0$. $\frac{\partial r}{\partial T} = -r \Rightarrow r = Ce^{-T}$. $\partial_T\phi = 0 \Rightarrow \phi = K$. $0 = C\cos(K)$ and $1 = -C\sin(K) \Rightarrow C = 1$ and $-\frac{\pi}{2} = K$. $x = Ce^{-T}\cos(\tau + K) = e^{-\epsilon t}\sin(t)$.

Problem 7.6.5

Analyze the two-timing system with $h = x\dot{x}^2$.
Solution: $r' = 0,\ \phi' = \frac{r^2}{8}$. Long-term behavior: $r(T) = r_0,\ \phi(T) = \frac{r_0^2}{8}T + \phi_0$. The origin is a linear center and the system is reversible, so the origin is a nonlinear center. The closed trajectories around the origin have an amplitude r_0 and frequency $\omega = 1 + \frac{r_0^2}{8}$.

Problem 7.6.15

Consider the pendulum equation $\ddot{x} + \sin x = 0$. Show that the frequency of small oscillations of amplitude $a \ll 1$ is given by $\omega \approx 1 - \frac{1}{16}a^2$.
Solution: Using the series expansion of sine, this becomes $\ddot{x} + x - \frac{x^3}{6} = 0$. Let $y = x/a$. Then $\ddot{y} + y - \frac{a^2}{6}y^3 = 0 = \ddot{y} + y + \epsilon y^3$, where $\epsilon = -\frac{a^2}{6}$. This can be mapped to the solution of Example 7.6.4, and $\omega = 1 + \frac{3}{8}\epsilon r^2 = 1 - \frac{1}{16}a^2$.

Problem 8.1.11

Find and classify all bifurcations for the system $\dot{x} = y - ax,\ \dot{y} = -by + \frac{x}{1+x}$.

Solution: The nullclines intersect at the origin and at $\left(\frac{1}{ab}-1, \frac{1}{b}-a\right)$. Linear analysis shows that $\tau = -(a+b)$ and $\Delta = ab-1$ for the origin, and $\tau = -(a+b)$ and $\Delta = ab-(ab)^2$ for the other fixed point. If we increase the value of ab from just below 1 to just above, the sign of Δ for the origin changes. Note also that when $ab = 1$ our second fixed point merges with the origin. As it passes by the origin it changes stability. Thus we conclude that along the curve $a = a/b$ we have transcritical bifurcations occuring. For both the origin and the other fixed point we find that $\tau^2 - \Delta$ is positive, so whenever Δ is positive that fixed point is a node and not a spiral.

Problem 8.1.11

Consider the system $\dot{u} = a(1-u) - uv^2,\ \dot{v} = uv^2 - (a+k)v$, where $a, k > 0$. Show that saddle-node bifurcations occur at $k = -a \pm \frac{1}{2}\sqrt{a}$.

Solution: $(1,0)$ is a solution for all a, k. Nullclines are given by $u = a/(a+v^2),\ u = (a+k)/v$. Solving for v at their intersection gives

$$v = \frac{a \pm \sqrt{a^2 - 4a(a+k)^2}}{2(a+k)}$$

To determine the bifurcation point analytically, the tangency condition gives $2av^3 = (a+k)(a+v^2)^2$. Equating the nullcline equations and plugging this in gives $a = v^2 \ \Rightarrow\ k = -a \pm \frac{\sqrt{a}}{2}$. Linear analysis shows that the fixed point at $(0,1)$ is a stable node. For the fixed points defined by the nullclines, linear analysis shows that we have a saddle node if $v^2 < a$, a nonhyperbolic fixed point if $v^2 = a$, and a stable node if $v^2 > a,\ k \leq a$. Therefore we have a saddle-node bifurcation at $k = -a \pm \frac{\sqrt{a}}{2}$.

Sample Problem 4c

$\dot{x} = \mu x + y - \sin x,\ \dot{y} = x - y$. $(0,0)$ is always a fixed point. Use a Taylor series to get $\dot{x} = \mu x + y - x + \frac{x^3}{6}$. Using the nullclines to solve for fixed points we get $x = \pm\sqrt{-6\mu}$, so $\mu_c = 0$. We have 1 fixed point for $\mu \geq 0$ and 3 for $\mu < 0$, so this is a pitchfork bifurcation (subcritical?).