Document: Math 201B
Latest Update: January 27, 2011
Author: Jeff Irion

Contents

- If $f'(x^*) < 0$ then the fixed point is stable. If $f'(x) > 0$ then the fixed point is unstable.
- The *potential*, $V(x)$, is defined by $f(x) = -\frac{dV}{dx}$

Saddle-Node Bifurcations

- $\dot{x} = r + x^2$

$$\dot{x} = f(x) = r + x^2$$
$$f'(x) = 2x$$

  3 Cases

  1. $r < 0 \Rightarrow 2$ fixed points (1 stable, 1 unstable)
     - $x = \sqrt{r} \Rightarrow f'(x) = 2\sqrt{r} > 0 \Rightarrow$ unstable
     - $x = -\sqrt{r} \Rightarrow f'(x) = -2\sqrt{r} < 0 \Rightarrow$ stable
  2. $r = 0 \Rightarrow 1$ half-stable fixed point at $x = 0$
  3. $r > 0 \Rightarrow$ no fixed points

  The bifurcation occurs at $r = r_c = 0$.

- $\dot{x} = r - x^2$

$$\dot{x} = f(x) = r - x^2$$
$$f'(x) = -2x$$

  3 Cases

  1. $r < 0 \Rightarrow$ no fixed points
  2. $r = 0 \Rightarrow 1$ half-stable fixed point at $x = 0$
  3. $r > 0 \Rightarrow 2$ fixed points (1 stable, 1 unstable)
     - $x = \sqrt{r} \Rightarrow f'(x) = -2\sqrt{r} < 0 \Rightarrow$ stable
     - $x = -\sqrt{r} \Rightarrow f'(x) = 2\sqrt{r} > 0 \Rightarrow$ unstable

  The bifurcation occurs at $r = r_c = 0$.

- $\dot{x} = r - x - e^{-x}$
  The system has fixed points at

$$r - x = e^{-x}$$

  We can solve this graphically by plotting $f(x) = r - x$ and $g(x) = e^{-x}$ and finding where the two curves intersect. The critical fixed point occurs when $r = r_c$ and the two curves are tangent to each other. We can solve for this by setting:

$$\frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x}$$
$$-1 = -e^{-x}$$
$$x^* = 0$$
$$r_c - 0 = e^{-0}$$
$$r_c = 1$$

Transcritical Bifurcations

- $\dot{x} = rx - x^2$

$$\dot{x} = f(x) = rx - x^2$$
$$f'(x) = r - 2x$$

3 Cases

1. $r < 0 \Rightarrow$ 2 fixed points (1 stable, 1 unstable)
   - $x = r \Rightarrow f'(x) = -r > 0 \Rightarrow$ unstable
   - $x = 0 \Rightarrow f'(x) = r < 0 \Rightarrow$ stable
2. $r = 0 \Rightarrow$ 1 half-stable fixed point at $x = 0$
3. $r > 0 \Rightarrow$ 2 fixed points (1 stable, 1 unstable)
   - $x = 0 \Rightarrow f'(x) = r > 0 \Rightarrow$ unstable
   - $x = r \Rightarrow f'(x) = -r < 0 \Rightarrow$ stable

Note that there is a fixed point at $x^* = 0$ for all values of r.
The bifurcation occurs at $r = r_c = 0$.

- $\dot{x} = x(1 - x^2) - a(1 - e^{-bx})$

$$\dot{x} = x(1 - x^2) - a(1 - e^{-bx})$$

Use a Taylor expansion for $1 - e^{-bx}$:

$$1 - e^{-bx} = bx - \frac{1}{2}b^2x^2 + O(x^3)$$

Plugging this into our ODE yields:

$$\dot{x} = (1 - ab)x + (\frac{1}{2}ab^2)x^2 + O(x^3)$$

The fixed points will occur at:

$$x^* = 0 \qquad \text{and} \qquad x^* \approx \frac{2(ab-1)}{ab^2}$$

Comparing to the normal form for a transcritical bifurcation, we see that the bifurcation occurs when $ab = 1$.

- $\dot{x} = r \ln x + x - 1$

$$\dot{x} = r \ln x + x - 1$$

Note that $x = 1$ is a fixed point for all values of r.
Introduce a new variable and use it to simplify the equation.

$$\begin{aligned} u &= x - 1 \\ \dot{u} &= \dot{x} \\ &= r\ln(1+u) + u \\ &= r\left[u - \frac{1}{2}u^2 + O(u^3)\right] + u \\ &\approx (r+1)u - \frac{1}{2}ru^2 \end{aligned}$$

Hence, a transcritical bifucation occurs at $r_c = -1$.
To put this equation into normal form, let

$$\begin{aligned} u &= a\nu \\ \dot{\nu} &\approx (r+1)\nu - \left(\frac{1}{2}ra\right)\nu^2 \\ &\approx (r+1)\nu - \nu^2 \qquad \text{where } a = \frac{2}{r} \\ &\approx \nu R - \nu^2 \qquad \text{where } R = r+1 \end{aligned}$$

In terms of our original variables, we have:

$$\begin{aligned} &R_c = 0 \Rightarrow r_c = -1 \\ &\nu^* = 0 \Rightarrow x^* = 1 \\ &\nu^* = R \Rightarrow u^* = \frac{2}{r}(r+1) \Rightarrow x^* = \frac{2r+2}{r} + 1 \end{aligned}$$

Pitchfork Bifurcations

- Supercritical: $\dot{x} = rx - x^3$

$$\begin{aligned} \dot{x} = f(x) &= rx - x^3 \\ f'(x) &= r - 3x^2 \end{aligned}$$

3 Cases

1. $r < 0 \Rightarrow 1$ stable fixed point at $x = 0$
2. $r = 0 \Rightarrow 1$ stable fixed point at $x = 0$
3. $r > 0 \Rightarrow 3$ fixed points (2 stable, 1 unstable)
   - $x = \sqrt{r} \Rightarrow f'(x) = -2r < 0 \Rightarrow$ stable
   - $x = 0 \Rightarrow f'(x) = r > 0 \Rightarrow$ unstable
   - $x = -\sqrt{r} \Rightarrow f'(x) = -2r < 0 \Rightarrow$ stable

---

- Bifurcation Overview

Saddle-Node	Transcritical	Pitchfork	
		Supercritical	Subcritical
$\dot{x} = r \pm x^2$	$\dot{x} = rx - x^2$	$\dot{x} = rx - x^3$	$\dot{x} = rx + x^3$

Saddle-node vs. transcritical: in the transcritical case, the two fixed points don't disappear after the bifurcation; instead, they just switch their stability.

Taylor Expansions

$$f(x) = f(x_0) + (x - x_0)\frac{\partial f}{\partial x}\Big|_{x_0} + \frac{1}{2!}(x - x_0)^2\frac{\partial^2 f}{\partial x^2}\Big|_{x_0} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \quad \text{for } |x| \leq 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \text{for all } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \quad \text{for } |x| < 1$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots \quad \text{for } -1 < x \leq 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \quad \text{for all } x$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \quad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \quad \text{for all } x$$

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots \quad \text{for } |x| < \frac{\pi}{2}$$