Stability

- $\mathbf{x}^*$ is an *attracting* fixed point if all trajectories that start near $\mathbf{x}^*$ approach it as $t \to \infty$
- $\mathbf{x}^*$ is *Liapunov stable* if all trajectories that start sufficiently close to $\mathbf{x}^*$ remain close to it for all time
  - A fixed point can be Liapunov stable but not attracting

Solving a Linear System

- $\tau = \text{trace}(A) = a + d = \lambda_1 + \lambda_2$
- $\Delta = \det(A) = ad - bc = \lambda_1 \lambda_2$
- $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$

Classification of Fixed Points

1. $\Delta < 0$: the eigenvalues are *real* and have opposite signs; the fixed point is a *saddle point*
2. $\Delta > 0$:
   (a) The eigenvalues are *real* with the same sign (*stable/unstable nodes*)
      i. $\lambda_1, \lambda_2 > 0 \;\Rightarrow$ unstable
      ii. $\lambda_1, \lambda_2 < 0 \;\Rightarrow$ stable
   (b) The eigenvalues are *complex*: $\lambda_{1,2} = \alpha \pm i\omega$
      i. $\alpha > 0 \;\Rightarrow$ *growing (unstable) spirals*
      ii. $\alpha = 0 \;\Rightarrow$ *circles*
      iii. $\alpha < 0 \;\Rightarrow$ *decreasing (stable) spirals*
      Note: direction (clockwise/counterclockwise) depends on initial conditions and must be checked.
   (c) $\tau^2 = 4\Delta \;\Rightarrow$ 1 eigenvalue
3. $\Delta = 0$: at least one of the eigenvalues is zero
   - The origin is not an isolated fixed point. There is either a whole line of fixed points, or a whole plane of fixed points if $A = 0$.

Existence and Uniqueness Theorem

Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Suppose that $\mathbf{f}$ is continuous and that all its partial derivatives $\partial f_i / \partial x_j$ are continous for $\mathbf{x}$ in some open connected set $D \subset \mathbb{R}^n$. Then for $\mathbf{x}_0 \in D$, the initial value problem has a solution $\mathbf{x}(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.
Corollary: Different trajectories never intersect.

Fixed Points and Linearization

Consider the system

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

with fixed point (x^*, y^*):

$$f(x^*, y^*) = 0$$
$$g(x^*, y^*) = 0.$$

Let

$$u = x - x^* \qquad v = y - y^*$$

We linearize about the fixed point (x^*, y^*) and get

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms}$$

If the linearized system predicts a saddle node, node, or a spiral, then the fixed point really *is* a saddle node, node, or spiral for the nonlinear system. In other words, if $\text{Re}(\lambda_1) \neq 0$ and $\text{Re}(\lambda_2) \neq 0$ then the linearization will give the correct result.

Theorem: (Nonlinear centers for reversible systems) Suppose the origin $\mathbf{x}^* = \mathbf{0}$ is a linear center for a continuously differentiable system and suppose that the system is reversible. Then sufficiently close to the origin, all trajectories are closed curves.

Theorem: (Nonlinear centers for conservative systems) Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and $\mathbf{f}$ is continuously differentiable. Suppose there exists a conserved quantity $E(\mathbf{x})$ and suppose that $\mathbf{x}^*$ is an isolated fixed point (i.e., there are no other fixed points in a small neighborhood surrounding $\mathbf{x}^*$). If $\mathbf{x}^*$ is a local minimum of E, then all trajectories sufficiently close to $\mathbf{x}^*$ are closed.

Polar Coordinates Identity

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

Index Theory

Define $\phi = \tan^{-1}(\dot{y}/\dot{x})$. The *index* of a closed curve C is defined as $I_C = \frac{1}{2\pi}[\phi]_C$, where $[\phi]_C$ is the net change in ϕ over one circuit, i.e. the number of counterclockwise revolutions made by the vector field as $\mathbf{x}$ moves once counterclockwise around C.

Properties of the Index

1. Suppose that C can be continuously deformed into C' without passing through a fixed point. Then $I_C = I_{C'}$.
2. If C doesn't enclose any fixed points, then $I_C = 0$.
3. If we reverse all the arrows in the vector field by changing $t \to -t$, the index is unchanged.
4. Suppose that the closed curve C is actually a *trajectory* for the system, i.e. C is a closed orbit. Then $I_C = +1$.

More Index Info

- The index of a fixed point, $\mathbf{x}^*$, is defined as I_C, where C is any closed curve that encloses $\mathbf{x}^*$ and no other fixed points.
  - $I = +1$ for a stable node, unstable node, spiral, center, degenerate node, and star
  - $I = -1$ for a saddle point
- **Theorem:** If a closed curve C surrounds n isolated fixed points, then $I_C = I_1 + I_2 + \ldots + I_n$.
- **Theorem:** Any closed orbit in the phase plane must enclose fixed points whose indices sum to $+1$.
- Closed orbits are impossible for the "rabbit vs. sheep" system.

Definitions

- Given an attracting fixed point $\mathbf{x}^*$, we define its *basin of attraction* to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$

- *Potential energy*, $V(x)$, is defined by $F(x) = -dV/dx$
  - $m\ddot{x} = F(x) \quad \Rightarrow \quad m\ddot{x} + \frac{dV}{dx} = 0 \quad \Rightarrow \quad E = \frac{1}{2}m\dot{x}^2 + V(x) = \text{constant}$
- Systems for which a conserved quantity exists are called *conservative systems*
  - A conservative system cannot have any attracting fixed points
- *Contours* are closed curves of constant energy
- *Homoclinic orbits* are trajectories that start and end at the same fixed point
- A system has *time-reversal symmetry* if its dynamics look the same whether time runs forward or backward
  - Any mechanical system of the form $m\ddot{x} = F(x)$ is symmetric under time reversal, i.e. the change of variables $t \to -t$
- A *reversible system* is any second-order system that is invariant under $t \to -t$, $y \to -y$
  - Any system of the form $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, where f is odd in y and g is even in y, is reversible
- *Heteroclinic trajectories* or *saddle connections* are pairs of trajectories that join twin saddle points

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Rabbits vs. Sheep

$$\begin{aligned} \dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - x - y) \\ x, y &\geq 0 \end{aligned}$$

$$\mathbf{x}^* = (0,0), \quad (0,2), \quad (3,0), \quad (1,1)$$

$$A = \begin{pmatrix} -2x + 3 - 2y & -2x \\ -y & -2y + 2 - x \end{pmatrix}$$

$$A\big|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad \lambda_1 = 3, \quad \lambda_2 = 2$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A\big|_{(0,2)} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \qquad \lambda_1 = -1, \quad \lambda_2 = -2$$

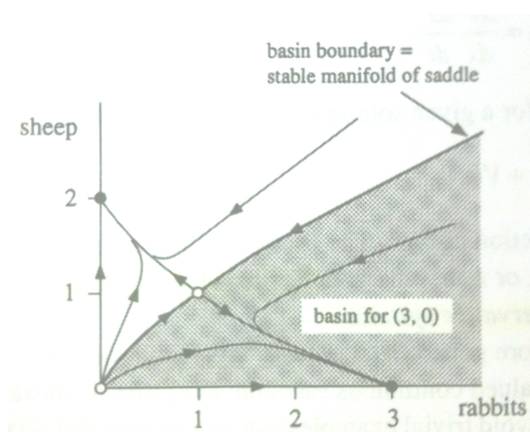
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A\big|_{(3,0)} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \qquad \lambda_1 = -1, \quad \lambda_2 = -3$$

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A\big|_{(1,1)} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \qquad \lambda_1 = -1 + \sqrt{2}, \quad \lambda_2 = -1 - \sqrt{2}$$

$$\mathbf{v}_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$



Conservative System

$$\text{Given:} \quad V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$

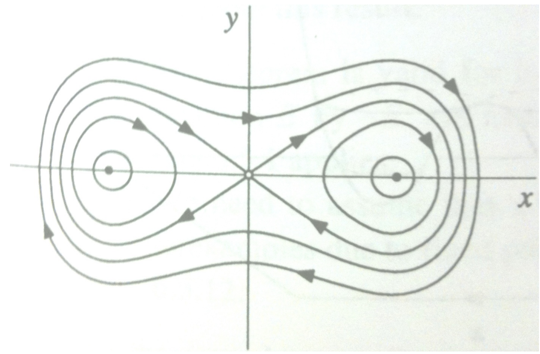
$$-\frac{dV}{dx} = x - x^3 = F(x)$$

We have fixed points at $(0,0)$ and $(\pm 1, 0)$. Linearize to get:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The trajectories are closed curves defined by the contours of constant energy, i.e.

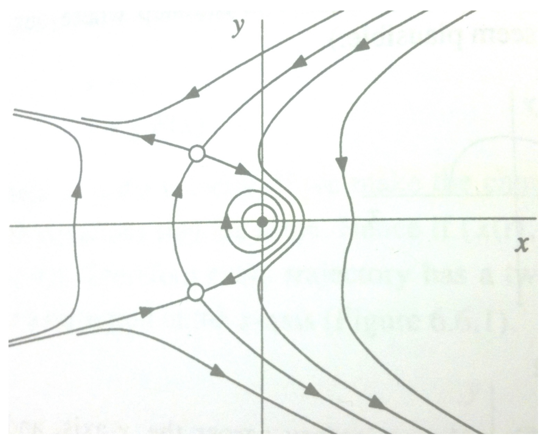
$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \text{constant}$$



Reversible System

$$\dot{x} = y - y^3$$
$$\dot{y} = -x - y^2$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



The origin is a fixed point with $\tau = 0$, $\Delta > 0$, so it is a linear center. The system is reversible, since the equations are invariant under the transformation $t \to -t$, $y \to -y$. Therefore, the origin is a *nonlinear center*. The system also has fixed points at $(-1, 1)$ and $(-1, -1)$, and they are saddle points. The twin saddle points are joined by a pair of trajectories. They are called *heteroclinic trajectories* or *saddle connections*.

Bifurcation Overview

Saddle-Node	Transcritical	Pitchfork	
		Supercritical	Subcritical
$\dot{x} = r \pm x^2$	$\dot{x} = rx - x^2$	$\dot{x} = rx - x^3$	$\dot{x} = rx + x^3$

Saddle-node vs. transcritical: in the transcritical case, the two fixed points don't disappear after the bifurcation; instead, they just switch their stability.

Taylor Expansions

$$f(x) = f(x_0) + (x - x_0)\frac{\partial f}{\partial x}\Big|_{x_0} + \frac{1}{2!}(x - x_0)^2\frac{\partial^2 f}{\partial x^2}\Big|_{x_0} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{for } |x| \le 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad \text{for } |x| < 1$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots \quad \text{for } -1 < x \le 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad \text{for all } x$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad \text{for all } x$$

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots \quad \text{for } |x| < \frac{\pi}{2}$$