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**Professor:** Thomases
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**Author:** Jeff Irion
`http://www.math.ucdavis.edu/~jlirion`

# Contents

# 1 Metric and Normed Spaces

## 1.1 Metrics and Norms

## 1.2 Convergence

**Definition 1.1.** ***Absolutely Convergent***
page 8

A series $\sum x_n$ is *absolutely convergent* if the sum of the absolute values $\sum |x_n|$ converges.

A useful property of an absolutely convergent series of real (or complex) numbers is that any series obtained from it by a permutation of its terms converges to the same sum as the original series.

**Example 1.2.** ***Absolute Convergence Example***
Problem 1.7 on page 30

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is not absolutely convergent because $\sum \frac{1}{n}$ does not converge. By rearranging terms, we can achieve two different sums:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \underbrace{-1 + \frac{1}{2}}_{<0} + \underbrace{-\frac{1}{3} + \frac{1}{4}}_{<0} + \underbrace{-\frac{1}{5} + \frac{1}{6}}_{<0} \dots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \left(-1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) + \left(-\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \dots\right) + \left(-\frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \dots\right) + \dots$$

The first sum is less than 0. The second sum is equal to 0, since each sum in parenthesis is 0.

## 1.3 Upper and Lower Bounds

**Definition 1.3.** $\limsup$ ***and*** $\liminf$
Definition 1.23 on page 10

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left[\sup\left\{x_k \middle| k \geq n\right\}\right]$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left[\inf\left\{x_k \middle| k \geq n\right\}\right]$$

## 1.4 Continuity

**Definition 1.4.** ***Upper Semicontinuous, Lower Semicontinuous***
page 14

A function is *upper semicontinuous* if

$$\limsup_{n\to\infty} f(x_n) \leq f(x).$$

A function is *lower semicontinuous* if

$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

## 1.5 Open and Closed Sets

**Proposition 1.5.**
Proposition 1.46 on page 17

$f : X \to Y$ is continuous iff $f^{-1}(G) \subset X$ is open $\forall\ G \subset Y$ open.

**Example 1.6.** ***Inverse Images of Closed Sets $\not\Rightarrow$ Continuity***

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\{1\}$ is closed in $\mathbb{R}$, but $f^{-1}\{1\} = (0, \infty)$ is not closed in $\mathbb{R}_{\geq 0}$

**Definition 1.7.** ***Urysohn's Lemma***
Exercise 1.16 on page 32 and Notes 9/29/10

Book
Suppose that $F$ and $G$ are closed and open subsets of $\mathbb{R}^n$, respectively, such that $F \subset G$. A continuous map $f : \mathbb{R}^n \to [0, 1]$ is given by

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}$$

Notes
Let $A$ and $B$ be closed disjoint subsets of a metric space $X$.

$$f(x) = \frac{d_B(x)}{d_A(x) + d_B(x)}$$

## 1.6 The Completion of a Metric Space

**Theorem 1.8.**
Theorem 1.52 on page 19

Every metric space has a completion. The completion is unique up to isomorphism.

## 1.7 Compactness

**Definition 1.9.** ***Sequentially Compact***
Definition 1.54 on page 23

A subset $K$ of a metric space $X$ is *sequentially compact* if every sequence in $K$ has a convergent subsequence whose limit belongs to $K$.

**Definition 1.10.** ***Totally Bounded***
Definition 1.58 on page 24

A subset of a metric space is *totally bounded* if it has a finite $\epsilon$-net for every $\epsilon > 0$.

**Theorem 1.11.**
Theorem 1.59 on page 25

A subset of a metric space is sequentially compact iff it is complete and totally bounded.

**Definition 1.12.** ***Compact***
Definition 1.60 on page 25

A subset $K$ of a metric space $X$ is *compact* if every open cover of $K$ has a finite subcover.

**Theorem 1.13.**
Theorem 1.62 on page 25

A subset of a metric space is compact iff it is sequentially compact.

**Theorem 1.14.** ***Heine-Borel***
Theorem 1.56 on page 23

A subset of $\mathbb{R}^n$ is sequentially compact iff it is closed and bounded.

**Theorem 1.15.** ***Bolzano-Weierstrass***
Theorem 1.57 on page 23

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

**Theorem 1.16.**
Theorem 1.66 on page 27

Images of compact sets are compact.

(more formally) Let $f : K \to Y$ be continuous on $K$, where $K$ is a compact metric space and $Y$ is any metric space. Then $f(K)$ is compact.

**Theorem 1.17.**
Theorem 1.67 on page 27

Let $f : K \to Y$ be continuous on a compact set $K$. Then $f$ is uniformly continuous.

## 1.8 Maxima and Minima

## 1.9 Chapter Summary

Everything in this chapter pertains to *metric* and/or *normed spaces*, with the main example being $\mathbb{R}^n$. We define *Cauchy sequences* and *convergent sequences*, which leads us to the concept of a *complete* metric space. We define *lim inf* and *lim sup*. We define *continuity* and ways to determine it (TFAE):

- $\epsilon$-$\delta$ definition
- $\lim\limits_{x \to x_0} f(x) = f(x_0)$
- If $x_n \to x$, then $f(x_n) \to f(x)$ (*sequential continuity*)
- For $G$ open, $f^{-1}(G)$ is open

We define an *open set* as one where each element can be contained in an open ball that lies entirely within the set. We define a *closed set* as a set whose complement is open $\Leftrightarrow$ a set that contains all its limit points.

The concepts of closed and complete are similar, but not the same. For example, the subset of rational numbers in $[0, 1]$ is closed in $\mathbb{Q}$ (not complete), but it is not closed (also, not open) in $\mathbb{R}$ (complete).

We define a *neighborhood* of a point as a set that contains an open set that contains that point. We prove that every metric space has a completion.

Finally, we define what it means for a subset of a metric space to be *compact* (TFAE):

- every open cover has a finite subcover
- every sequence has a convergent subsequence (*sequentially compact*)
- it is complete and totally bounded

The *Bolzano-Weierstrass Theorem* tells us that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. From this, the *Heine-Borel Theorem* tells us that a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded. We define what it means for a collection of subsets to be a *cover* and for a collection of points to be an $\epsilon$*-net.* If for every $\epsilon > 0$ a subset has a finite $\epsilon$-net, we say that it is *totally bounded.* Finally, we prove that continuous functions on compact domains attain their maximum and minimum.

# 2 Continuous Functions

## 2.1 Convergence of Functions

## 2.2 Spaces of Continuous Functions

**Theorem 2.1.**
Theorem 2.4 on page 38

Let $K$ be a compact metric space. The space $C(K)$ is complete wrt the uniform norm.

**Definition 2.2.** ***Support***
Definition 2.6 on page 39

The *support*, supp $f$, of a function $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is the closure of the set on which $f$ is nonzero.

$$\text{supp } f = \overline{\{x \in X \mid f(x) \neq 0\}}$$

We say that $f$ has *compact support* if supp $f$ is a compact subset of $X$.

**Definition 2.3.** ***Spaces of Continuous Functions***
page 39

$$C(X) \supset C_b(X) \supset C_0(X) \supset C_c(X)$$

- $C_c(X)$ = the space of continuous functions on $f$ with compact support
- $C_0(X)$ = the closure of $C_c(X)$ = the functions that vanish at infinity
- $C_b(X)$ = the space of bounded continuous functions on $X$
- $C(X)$ = the space of continuous functions on $X$

If $X$ is compact then these spaces are equal.

## 2.3 Approximation by Polynomials

**Theorem 2.4.** ***Weierstrass Approximation Theorem***
Theorem 2.9 on page 40

The set of polynomials is dense in $C([a, b])$.

**Theorem 2.5.** ***Stone-Weierstrass Theorem***
`http://en.wikipedia.org/wiki/Stone-Weierstrass_theorem#Stone.E2.80.93Weierstrass_theorem.2C_real_version`

Suppose $X$ is a compact Hausdorff space and $A$ is a subalgebra of $C(X, R)$ which contains a non-zero constant function. Then $A$ is dense in $C(X, R)$ if and only if it separates points.

### 2.4 Compact Subsets of $C(K)$

**Definition 2.6.** ***Equicontinuous***
Definition 2.10 on page 44

Let $\mathcal{F}$ be a family of functions from a metric space $(X, d)$ to a metric space $(Y, d)$. The family $\mathcal{F}$ is *equicontinuous* if for every $x \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$ for all $f \in \mathcal{F}$.

Note: $\delta$ may depend on $x$, but it does not depend on $f$.

**Theorem 2.7.** ***Arzelà-Ascoli***
Theorem 2.12 on page 45

Let $K$ be a compact metric space. A subset of $C(K)$ is compact iff it is closed, bounded, and equicontinuous.

**Definition 2.8.** ***Lipschitz Continuous***
Definition 2.15 on page 48

A function $f : X \to R$ on a metric space $X$ is *Lipschitz continuous* if there is a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq Cd(x, y) \quad \forall\ x, y \in X$$

The smallest such constant $C$ is denoted $\mathrm{Lip}(f)$.

### 2.5 Chapter Summary

This chapter focuses on spaces of continuous functions equipped with the *uniform norm*. An important result is that if a sequence of continuous functions $(f_n)$ converges uniformly to a function $f$, then $f$ is continuous. Similarly, for $K$ compact, $C(K)$ is complete. The *Weierstrass Approximation Theorem* tells us that the set of polynomials is dense in $C([a, b])$. We define what it means for a family of functions $\mathcal{F}$ to be *equicontinuous.*

In Chapter 1, the Heine-Borel Theorem told us that a subset of a finite-dimensional normed space is compact iff it is closed and bounded. Here, we develop the *Arzelà-Ascoli Theorem* which characterizes the compact subsets of $C(K)$: $\mathcal{F} \subset C(K)$ is compact iff it is closed, bounded, and equicontinuous. Finally, we define what it means for a function to be *Lipschitz continuous*: $\exists\ C \geq 0$ such that

$$|f(x) - f(y)| \leq Cd(x, y) \qquad \forall\ x, y.$$

# 3 The Contraction Mapping Theorem

## 3.1 Contractions

**Definition 3.1.** ***Contraction***
Definition 3.1 on page 61

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists a constant $c$, with $0 \le c < 1$, such that

$$d(Tx, Ty) \le cd(x, y) \quad \forall\, x, y \in X$$

**Theorem 3.2.** ***Contraction Mapping Theorem***
Theorem 3.2 on page 62

If $T : X \to X$ is a contraction mapping on a complete metric space, then there is exactly one solution $x \in X$ of

$$Tx = x$$

## 3.2 Fixed Points of Dynamical Systems

## 3.3 Integral Equations

## 3.4 Boundary Value Problems for Differential Equations

## 3.5 Initial Value Problems for Differential Equations

## 3.6 Chapter Summary

This chapter is pretty much summed up by the *Contraction Mapping Theorem.*

# 4 Topological Spaces

## 4.1 Topological Spaces

**Definition 4.1.** ***Topology***
Definition 4.1 on page 81

A *topology* is a collection $\mathcal{T}$ of subsets of $X$ such that

(a) $\emptyset, X \in \mathcal{T}$
(b) If $G_\alpha \in \mathcal{T}$ for $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha \in \mathcal{T}$
(c) If $G_i \in \mathcal{T}$ for $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^{n} G_i \in \mathcal{T}$

**Definition 4.2.** ***Indescrete and Discrete Topology***
Example 4.2 on page 81

- $\mathcal{T} = \{\emptyset, X\}$ is called the *indescrete topology*
- $\mathcal{T} = \mathcal{P}(X)$ (the power set of $X$) is called the *discrete topology*

**Definition 4.3.** ***Induced Topology***
Example 4.4 on page 82

Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. We define the *induced topology*, $\mathcal{S}$, as

$$\mathcal{S} = \left\{ H \subset Y \;\middle|\; H = G \cap Y \text{ for some } G \in \mathcal{T} \right\}$$

**Definition 4.4.** ***Hausdorff***
page 82

A topology is *Hausdorff* if for every pair of distinct points $x$ and $y$ there exist neighborhoods $V_x$ and $V_y$ such that $x \in V_x$, $y \in V_y$, and $V_x \cap V_y = \emptyset$.

**Definition 4.5.** ***Convergence***
Definition 4.5 on page 82

A sequence $(x_n)$ *converges* to a limit $x \in X$ if for every neighborhood $V$ of $x$, there is a number $N$ such that $x_n \in V \;\forall\; n \geq N$.

**Definition 4.6.** ***Continuous***
Definition 4.6 on page 83

A function $f : X \to Y$ is *continuous* at $x \in X$ if for each neighborhood $W \subset Y$ of $f(x)$ there exists a neighborhood $V$ of $x$ such that $f(V) \subset W$

**Theorem 4.7.**
Theorem 4.7 on page 83

Given: $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces and $f : X \to Y$
$f$ is continuous on $X$ iff $f^{-1}(G) \in \mathcal{T}$ for every $G \in \mathcal{S}$

**Definition 4.8.** ***Compact***
Definition 4.10 on page 83

A subset $K$ of a topological space is *compact* if every open cover of $K$ contains a finite subcover.

## 4.2 Bases of Open Sets

**Definition 4.9.** ***Base, Neighborhood Base, First Countable, Second Countable***
Definition 4.11 on page 84

- A subset $\mathcal{B}$ of a topology $\mathcal{T}$ is a *base* for $\mathcal{T}$ if for every $G \in \mathcal{T}$ there is a collection of subsets $B_\alpha \subset \mathcal{B}$ such that $G = \bigcup_\alpha B_\alpha$
- A collection $\mathcal{N}$ of neighborhoods of a point $x \in X$ is called a *neighborhood base* for $x$ if for each neighborhood $V$ of $x$ there is a neighborhood $W \in \mathcal{N}$ such that $W \subset V$
- A topological space is *first countable* if every $x \in X$ has a countable neighborhood base
- A topological space is *second countable* if $X$ has a countable base

## 4.3 Comparing Topologies

**Definition 4.10.** ***Finer/Stronger and Coarser/Weaker***
Definition 4.11 on page 84

If $\mathcal{T}_1 \subset \mathcal{T}_2$, then

- $\mathcal{T}_2$ is *finer* or *stronger* that $\mathcal{T}_1$
- $\mathcal{T}_1$ is *coarser* or *weaker* than $\mathcal{T}_2$

## 4.4 Chapter Summary

This chapter generalizes the concepts in chapter 1 to topological spaces, which is to say that it defines them in a way that depends only on open and closed sets.

- A sequence $(x_n)$ *converges* to $x$ if for every neighborhood $V_\epsilon$ of $x$ there exists $N$ such that $n \geq N$ implies $x_n \in V_\epsilon$.
- A function is *continuous* at $x$ if for every neighborhood $W_\epsilon$ of $f(x)$ there exists a neighborhood $V_\delta$ such that $f(V_\delta) \subset W_\epsilon$.
  - $f$ is continuous iff $f^{-1}(G)$ is open for every $G$ open.

- A subset is *compact* if every open cover contains a finite subcover.

We introduce a method for comparing different topologies of the same metric space, and we use this concept to prove several continuity results.

# 5 Banach Spaces

## 5.1 Banach Spaces: Definition and Examples

**Definition 5.1.** ***Banach Space***
Definition 5.1 on page 91

A *Banach space* is a complete normed linear space.

**Definition 5.2.** ***Hölder's Inequality***
Notes 11/3/10

$$\sum |x_i||y_i| \leq \|x\|_p \|y\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

**Definition 5.3.** ***Minkowski's Inequality***
Notes 11/3/10

$$\|x + y\|_p \leq \|x\|_p + \|y\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

**Example 5.4.** ***Sequence Spaces***
Example 5.5 on page 92

For $1 \leq p < \infty$, the *sequence space* $\ell^p(\mathbb{N})$ consists of all infinite sequences such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

with the $p$-norm,

$$\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

$\ell^p(\mathbb{N})$ is an infinite-dimensional Banach space for $1 \leq p \leq \infty$.

**Example 5.5.** ***$L^p([a,b])$ and the $L^p$-norm***
Example 5.6 on page 92

$$L^p([a,b]) = \left\{ f \;\middle|\; f : [a,b] \to \mathbb{R} \text{ (or } \mathbb{C}), \int_a^b |f(x)|^p \, dx < \infty \right\}$$
$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}$$

Note: $\Omega$ is used to denote a measureable subset of $\mathbb{R}^n$, which could be $\mathbb{R}^n$ itself. Then $L^p(\Omega)$ is the set of Legesgue measurable functions $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) whose $p$th power is Lebesgue integrable, with the norm $\|f\|_p$

## 5.2 Bounded Linear Maps

**Definition 5.6.** ***Linear Map***
page 95

A *linear map* is a function between real (or complex) linear spaces $X$, $Y$ such that

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty$$

**Definition 5.7.** ***Bounded***
Definition 5.12 on page 95

A linear map $T : X \to Y$ is *bounded* iff $\exists\ M \geq 0$ such that

$$\|Tx\| \leq M \|x\| \;\; \forall x \;\in X$$

and we define the *uniform norm* of $T$ as

$$\|T\| = \inf \left\{ M \;\middle|\; \|Tx\| \leq M \|x\| \;\; \forall\ x \in X \right\}$$

We denote the set of all linear maps from $X$ to $Y$ by $\mathcal{L}(X, Y)$.
We denote the set of all bounded linear maps from $X$ to $Y$ by $\mathcal{B}(X, Y)$.

Equivalent expressions for $\|T\|$:

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}; \qquad \|T\| = \sup_{\|x\| \leq 1} \|Tx\|; \qquad \|T\| = \sup_{\|x\| = 1} \|Tx\|$$

A linear map $A : X \to Y$ between finite-dimensional real linear spaces with dim $X = n$, dim $Y = m$ may be represented as a matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**Theorem 5.8.**
Theorem 5.18 on page 99 and Notes 11/5/10

A linear map is bounded iff it is uniformly continuous.

**Corollary:** If a linear map is continuous at zero, then it is uniformly continuous everywhere.

Proof
**Given:** $T$ bounded $\Rightarrow \exists\, M$ such that $\|Tx - Ty\| = \|T(x-y)\| \leq M\|x-y\|$
**Want:** $T$ continuous

Fix $\epsilon > 0$. Let $\delta = \epsilon/M$. Then $\|x-y\| < \delta$ implies $\|Tx - Ty\| < \epsilon$, and therefore $T$ is uniformly continuous.

**Given:** $T$ continuous
**Want:** $T$ bounded

$T$ is continuous at 0, so $\exists\, \delta > 0$ such that $\|x - 0\| = \|x\| < \delta$ implies $\|Tx - T0\| = \|T(x-0)\| = \|Tx\| < 1$. For any $x \in X$, with $x \neq 0$, define $\tilde{x}$ by

$$\tilde{x} = \delta \frac{x}{\|x\|}$$

Then $\|\tilde{x}\| < \delta$, so $\|T\tilde{x}\| < 1$. Then

$$\|Tx\| = \frac{\|x\|}{\delta}\|T\tilde{x}\| \leq M\|x\|$$

so T is bounded.

**Proposition 5.9.**
Notes 11/5/10

$\mathcal{B}(X)$ is a normed linear space, with $\|T\| = \sup_{\|x\|=1} \|Tx\|$

Proof Outline

- Show that $\mathcal{B}(X)$ is a linear space
- Check that it satisfies the properties of a norm
- Show that $S, T \in \mathcal{B}(X)$ implies $ST \in \mathcal{B}(X)$ (See Theorem 5.27)

**Theorem 5.10.**
Notes 11/5/10

Let $X$ be a normed linear space and $Y$ a Banach space. $\mathcal{B}(X, Y)$ is a Banach space.
(Appears again as Theorem 5.29)

Proof Outline

- Let $(T_n)$ be Cauchy in $\mathcal{B}(X, Y)$.
- Use boundedness to show that $(T_n(x))$ is Cauchy in $Y$.

- Define $Tx = \lim_{n\to\infty} T_n(x)\ \forall x \in X$
- Show $T$ is linear and bounded, and that $T_n \to T$ wrt the operator norm.

**Theorem 5.11.** ***Bounded Linear Transformation (BLT) Theorem***
Theorem 5.19 on page 100

Let $X$ be a normed linear space and $Y$ a Banach space. If $M$ is a dense linear subspace of $X$ and

$$T : M \subset X \to Y$$

is a bounded linear map, then $\exists$ a unique bounded linear map $\overline{T} : X \to Y$ such that $\overline{T}x = Tx\ \forall\ x \in M$, and $\|\overline{T}\| = \|T\|$.

Compare to the Hahn-Banach Theorem (5.36).

Proof Outline

- For every $x \in X$ there is a sequence $(x_n) \in M,\ x_n \to x$. Define

$$\overline{T}x = \lim_{n\to\infty} Tx_n$$

- Use $T$ bounded, $x_n$ Cauchy, and $Y$ complete to show that this limit exists and is unique.
- Show that $\overline{T}$ is bounded, and that $\|\overline{T}\| = \|T\|$
- Uniqueness: if $\tilde{T}$ is another such map, show that $\tilde{T} = \overline{T}$

**Definition 5.12.** ***Equivalence of Norms***
Definition 5.21 on page 101

Two norms on $X$ are equivalent if $\exists\ c > 0$ and $C > 0$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1\ \forall\ x \in X$$

**Definition 5.13.** ***Strength of Norms***
Notes 11/5/10

$\|\cdot\|_2$ is *stronger* than $\|\cdot\|_1$ if

$$\|x_n\|_2 \to 0 \Rightarrow \|x_n\|_1 \to 0$$

Equivalently, $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$ iff $\exists\ c > 0$ such that

$$\|x\|_1 \leq c\|x\|_2\ \forall\ x \in X$$

**Theorem 5.14.**
Notes 11/5/10

Two norms on a linear space generate the same topology iff they are equivalent.

**Definition 5.15.** ***Open Mapping***
Notes 11/10/10

$T : X \to Y$ is an *open mapping* if $T(U) \subset Y$ is open whenever $U \subset X$ is open.

**Theorem 5.16.** ***Open Mapping Theorem***
Theorem 5.23 on page 101 and Notes 11/10/10 & 11/12/10

If $T \in \mathcal{B}(X, Y)$, $X$ and $Y$ are Banach spaces, and $T$ is onto, then $T$ is an open mapping.
If $T$ is one-to-one, then $T^{-1} \in \mathcal{B}(X, Y)$.

**Theorem 5.17.**
Notes 11/12/10

If $T \in \mathcal{B}(X, Y)$, $X$ and $Y$ are Banach spaces, then TFAE

(a) $T$ has closed range and is one-to-one
(b) $\exists\, c > 0$ such that $c\|x\| \le \|Tx\| \quad \forall x \in X$

(Related to Theorems 5.19 and 5.23)

Proof Outline
(a) $\Rightarrow$ (b)

- $T : X \to \text{ran } T$ is one-to-one and onto
- By the Open Mapping Theorem, $T^{-1} : \text{ran } T \to X$ is bounded
- ran $T$ is a Banach space because $T$ has closed range
- $y \in \text{ran } T,\ \|x\| = \|T^{-1}y\| \le c_1\|y\| = c_1\|Tx\|$

(b) $\Rightarrow$ (a)

- $Tx = 0 \Rightarrow x = 0 \Rightarrow \ker T = 0 \Rightarrow T$ is one-to-one
- Pick $(y_n)$ convergent in ran $T \Rightarrow y_n \to y \in Y$
- There is a corresponding sequence $(x_n) \in X$
- Use (b) to show that $(x_n)$ is Cauchy in $X \Rightarrow x_n \to x \in X$
- $Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y \Rightarrow y \in \text{ran } T$

## 5.3 The Kernel and Range of a Linear Map

**Definition 5.18.** ***Kernel and Range***
Definition 5.24 on page 102

$$\ker T = \{x \in X \mid Tx = 0\}$$
$$\text{ran } T = \{y \in Y \mid \exists\, x \in X \text{ such that } Tx = y\}$$

**Proposition 5.19.**
Notes 11/10/10

$T$ is one-to-one iff $\ker T = \{0\}$
$T$ is onto iff $\text{ran } T = Y$

(Related to Theorems 5.17 and 5.23)

**Theorem 5.20.**
Theorem 5.25 on page 102 and Notes 11/10/10

Given: $T : X \to Y$, a linear map between linear spaces
The kernel of $T$ is a linear subspace of $X$.
The range of $T$ is a linear subspace of $Y$.
If $X$ and $Y$ are normed linear spaces and $T$ is bounded, then $\ker T$ is a closed linear subspace of $X$.

Proof Outline
$x_1,\ x_2 \in \ker T \Rightarrow \lambda_1 x_1 + \lambda_2 x_2 \in \ker T$
$y_1,\ y_2 \in \text{ran } T \Rightarrow \lambda_1 y_1 + \lambda_2 y_2 \in \text{ran } T$
Closed: If $(x_n) \in \ker T$ and $(x_n) \to x$, continuity implies that

$$Tx = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = 0$$

**Definition 5.21.** ***Nullility and Rank***
page 102

Nullility := the dimension of $\ker T$
Rank := the dimension of $\text{ran } T$

**Example 5.22.** ***Left and Right Shift Operators***
Example 5.26 on page 103

Define the right shift operator as:

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$$

Define the left shift operator as:

$$T(x_1, x_2, x_3 \ldots) = (x_2, x_3, \ldots)$$

The right shift operator $S$ is one-to-one but not onto.
The left shift operator $T$ is onto but not one-to-one.

**Proposition 5.23.**
Proposition 5.30 on page 105

TFAE:

(a) $\exists\, c > 0$ such that
$$c\|x\| \le \|Tx\| \quad \forall x \in X$$

(b) $T$ has closed range, and the only solution of $Tx = 0$ is $x = 0$

(Related to Theorems 5.17 and 5.19)

Proof Outline

**(a) $\Rightarrow$ (b)**

- $Tx = 0$ implies $\|x\| = 0 \Rightarrow x = 0$
- Given a sequence $(y_n) \to y \in Y$, there is a sequence $(x_n)$ in $X$ such that $Tx_n = y_n$
- Show that this sequence is Cauchy, so $x_n \to x \in X$, and use boundedness to show that $Tx = y$

**(b) $\Rightarrow$ (a)**

- ran $T$ is closed and therefore a Banach space, so $T : X \to \text{ran } T$ is a one-to-one, onto map between Banach spaces
- By the open mapping theorem (Theorem 5.16 in pdf, Theorem 5.23 in book), $T^{-1}$ is bounded, and this implies (a).

## 5.4 Finite-Dimensional Banach Spaces

**Theorem 5.24.**
Theorem 5.36 on page 108

Any two norms on a finite-dimensional space are equivalent.

Proof Outline

- Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be 2 norms on $X$

- Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of $X$
- $\exists\; m_1, m_2, M_1, M_2 \in \mathbb{R}_{>0}$ such that

$$m_1 \sum_{i=1}^{n} |x_i| \leq \|x\|_1 \leq M_1 \sum_{i=1}^{n} |x_i|$$
$$m_2 \sum_{i=1}^{n} |x_i| \leq \|x\|_2 \leq M_2 \sum_{i=1}^{n} |x_i|$$

- It follows that $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$

**Theorem 5.25.**
Theorem 5.33 on page 107 and Notes 11/8/10

Every finite-dimensional normed linear space is a Banach space.
**Corollary:** Every finite-dimensional linear subspace of a normed linear space is closed.

Proof Outline
Completeness follows from the fact that $(X, \|\cdot\|_{\text{Euclidean}})$ is complete.

**Theorem 5.26.**
Theorem 5.35 on page 107 and Notes 11/8/10

Every linear operator on a finite-dimensional linear space is bounded.

Proof Outline

- Let $A : X \to Y,\; A \in \mathcal{L}(X, Y),\; \dim(X) = n$
- Let $\{e_1, \ldots, e_n\}$ be a basis for $X$. Then $x = \sum_{i=1}^{n} x_i e_i,\; x_i \in \mathbb{R}$, and

$$\|Ax\| \leq \sum_{i=1}^{n} |x_i| \|Ae_i\| \leq \max_{1 \leq i \leq n} \{\|Ae_i\|\} \sum_{i=1}^{n} |x_i| \leq \frac{1}{m} \max_{1 \leq i \leq n} \{\|Ae_i\|\} \|x\|$$

## 5.5 Convergence of Bounded Operators

**Theorem 5.27.**
Theorem 5.37 on page 108

Let $X$, $Y$, and $Z$ be normed linear spaces. If $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$, then $ST \in \mathcal{B}(X, Z)$, and

$$\|ST\| \leq \|S\|\|T\|$$

**Definition 5.28.** ***Uniform Convergence of Operators***
Definition 5.39 on page 109

Given $(T_n) \subset \mathcal{B}(X,Y)$. $T_n$ *converges uniformly* to $T \in \mathcal{B}(X,Y)$ if

$$\lim_{n\to\infty} \|T_n - T\| = 0$$

See Convergence Overview (Example 5.51).

**Theorem 5.29.**
Theorem 5.41 on page 110

If $X$ is a normed linear space and $Y$ is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space with respect to the operator norm.
(Seen before as Theorem 5.10)

Proof Outline

- To prove that $\mathcal{B}(X,Y)$ is complete, let $(T_n)$ be a Cauchy sequence in $\mathcal{B}(X,Y)$
- $\|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\| \Rightarrow (T_n x)$ is Cauchy in $Y$
- Since $Y$ is complete, $(T_n x) \to y \in Y$
- Define $Tx = \lim_{n\to\infty} T_n x$
- $T$ Cauchy + $\triangle$ Inequality $\Rightarrow T$ bounded and $\lim_{n\to\infty} \|T_n - T\| = 0$, so $\mathcal{B}(X,Y)$ is complete

**Definition 5.30.** ***Compact Operator***
Definition 5.42 on page 110 and Notes 11/17/10

$T : X \to Y$ is a *compact operator* if $T(B)$ is a precompact subset of $Y$ for every bounded subset $B \subset X$.

Equivalently, $T$ is a compact operator if $\forall\ (x_n) \in X$, $(x_n)$ bounded, there exists a subsequence $(x_{n_k})$ such that $T(x_{n_k}) \to y \in Y$.

**Definition 5.31.** ***Strong Convergence***
Definition 5.44 on page 111

A sequence $(T_n) \subset \mathcal{B}(X,Y)$ *converges strongly* if

$$\lim_{n\to\infty} T_n x = Tx \quad \forall\ x \in X$$

For linear maps, strong convergence means pointwise convergence.

See Convergence Overview (Example 5.51).

**Theorem 5.32.**
Theorem 5.45 on page 111

If $T_n \to T$ uniformly, then $T_n \to T$ strongly.

**Example 5.33.** ***Strong Convergence $\not\Rightarrow$ Uniform Convergence***
Example 5.46 on page 111

Let $X = \ell^p(\mathbb{N})$ and define $P_n : X \to X$ by

$$P_n(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

Then $\|P_n - P_m\| = 1 \;\forall\; n \neq m$, so $(P_n)$ does not converge uniformly. However, $P_n x \to x$ as $n \to \infty$, so $P_n \to I$ strongly.

See Convergence Overview (Example 5.51).

## 5.6 Dual Spaces

**Definition 5.34.** ***Linear Functional***
Notes 11/12/10

Let $X$ be a normed linear space. $T \in \mathcal{L}(X, \mathbb{R})$ is called a *linear functional* on X.

**Definition 5.35.** ***Dual Space***
Notes 11/12/10

If $X$ is a normed linear space, then $\mathcal{B}(X, \mathbb{R}) = X^*$ is a Banach space called the *dual space* of X.

**Theorem 5.36.** ***Hahn-Banach Theorem***
Theorem 5.58 on page 118 and Notes 11/12/10

**Given:** $Y$ is a linear subspace of a normed linear space $X$. $\psi : Y \to \mathbb{R}$ is a bounded linear functional on $Y$ with $\|\psi\| = M$.
**Then:** There is a bounded linear functional $\varphi : X \to \mathbb{R}$ on $X$ such that $\varphi$ restricted to $Y$ is equal to $\psi$ and $\|\phi\| = M$.

**Corollary:** $\forall\; x \in X,\; x \neq 0,\; \exists\; \varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$
**Corollary:** $\forall\; x, y \in X,$ if $\varphi(x) = \varphi(y) \;\forall\; \varphi \in X^*$ then $x = y$
**Corollary:** If $x_0 \in X,\; x_0 = 0$ iff $\varphi(x_0) = 0 \;\forall\; \varphi \in X^*$

Compare to the Bounded Linear Transformation (BLT) Theorem (5.11).

**Theorem 5.37.** ***Riesz Representation Theorem***
Notes 11/15/10

Given $\varphi \in \mathcal{B}(\mathcal{H}, \mathbb{R})$, i.e. $\varphi \in \mathcal{H}^*$, i.e. $\varphi$ is a linear functional on $\mathcal{H}$. Then there is a unique $y \in \mathcal{H}$ such that $\varphi(x) = \langle y, x \rangle \ \forall\, x \in \mathcal{H}$. In other words, every linear functional in $\mathcal{H}$ can be identified with an inner product with $y \in H$.

**Example 5.38.** ***Applications of the Riesz Representation Theorem***
Notes 11/15/10

$(L^p)^* = L^q$
$(L^2)^* = L^2$

**Definition 5.39.** ***Bidual Space***
page 118

Since $X^*$ is a Banach space (See Theorem 5.29), we can form its dual space, $X^{**}$, called the *bidual space* of $X$.

We can identify an element $x \in X$ with an element $F_x \in X^{**}$ by:

$$F_x(\varphi) = \varphi(x) \quad \forall\, \varphi \in X^*$$

Thus, we may regard $X$ as a subspace of $X^{**}$.

**Definition 5.40.** ***Reflexive***
Notes 11/12/10 and page 119

If $X = X^{**}$ under the identification $x \mapsto F_x$ then $X$ is *reflexive*.

Example: $\ell^p = \ell^{p**}$

**Definition 5.41.** ***Weak Convergence***
Definition 5.59 on page 119

A sequence $(x_n)$ in a Banach space $X$ *converges weakly* to $x$, denoted by $x_n \rightharpoonup x$ as $n \to \infty$, if

$$\varphi(x_n) \to \varphi(x) \quad \text{as } n \to \infty \quad \forall\, \varphi \in X^*$$

If we think of a linear functional $\varphi : X \to \mathbb{R}$ as defining a coordinate $\varphi(x)$ of $x$, then weak convergence in a finite dimension corresponds to coordinate-wise convergence.

See Convergence Overview (Example 5.51).

**Example 5.42.** ***Motivation for Strong and Weak Convergence***
Notes 11/15/10

In finite dimensions, every bounded sequence has a convergent subsequence. However, this is not the case in infinite dimensions.

$$X = \ell^\infty, \qquad e_n = (0, 0, 0, \ldots, 0, 1, 0, 0, \ldots)$$
$$n^{\text{th}} \text{ position} \uparrow$$

$(e_n)$ is a bounded sequence with no convergent subsequence.

See Convergence Overview (Example 5.51).

**Example 5.43.** ***Uniform, Strong, and Weak Convergence***
Notes 11/17/10

Consider $\ell^2$, the left shift operator $T$, and the right shift operator $S$ (See Example 5.22).
Let $(S_n) = S^n$, $(T_n) = T^n$

$T_n \to 0$ strongly, but $T_n \not\rightrightarrows 0$ uniformly.
$S_n \rightharpoonup 0$ weakly, but $S_n \not\to 0$ strongly.

See Convergence Overview (Example 5.51).

**Proposition 5.44.**
Notes 11/15/10

1. If $(x_n)$ is weakly convergent in $X$, then the weak limit is unique.
2. If $x_n \to x \in X$ then $x_n \rightharpoonup x$

**Proposition 5.45.**
Notes 11/17/10

If $\dim(X) = \infty$ then the weak topology is not metrizable.
If $\dim(X) < \infty$ then the weak topology coincides with the strong topology.

**Definition 5.46.** ***Weak-$*$ Convergence***
Definition 5.60 on page 119

Let $X^*$ be the dual space of a Banach space $X$. We say that $\varphi \in X^*$ is the *weak-$*$* limit of a sequence $(\varphi_n) \subset X^*$ if

$$\varphi_n(x) \to \varphi(x) \quad \text{as } n \to \infty \quad \forall\, x \in X$$

We denote weak-$*$ convergence by

$$\varphi_n \overset{*}{\rightharpoonup} \varphi$$

Note: by contrast, weak convergence of $(\varphi_n)$ in $X^*$ means that

$$F(\varphi_n) \to F(\varphi) \quad \text{as } n \to \infty \quad \forall\, F \in X^{**}$$

If $X = X^{**}$, i.e. $X$ is reflexive, then weak convergence = weak-$*$ convergence.

See Convergence Overview (Example 5.51).

**Theorem 5.47.** ***Banach-Alaoglu Theorem***
Theorem 5.61 on page 119

Let $X^*$ be the dual space of a Banach space $X$. The closed unit ball in $X^*$ is weak-$*$ compact.

**Theorem 5.48.**
Notes 11/17/10

If $X$ is a separable Banach space then any bounded sequence $(f_n) \in X^*$ has a weak-$*$ convergent subsequence.

**Theorem 5.49.**
Notes 11/17/10

If $X$ is reflexive then the closed unit ball in $X$ is weak sequentially compact.

**Example 5.50.** ***Convergence of Bounded Operators***
Notes 11/17/10

$T_n, T \in \mathcal{B}(X, Y)$

1. $T_n$ converges uniformly to $T$ if $\|T_n - T\| \to 0$, i.e. $T_n \to T$ in the operator norm, denoted $T_n \rightrightarrows T$.
2. $T_n$ converges strongly to $T$, $T_n \to T$, if $\forall\, x \in X,\ T_n x \to Tx$, i.e. $\|(T_n - T)(x)\|_Y \to 0$
3. $T_n \rightharpoonup T$ weakly if $\forall\, x \in X,\ T_n x \rightharpoonup Tx \in Y$, i.e. $\forall\, f \in Y^*,\ f(T_n x) \to f(Tx)$.

See Convergence Overview (Example 5.51).

**Example 5.51.** ***Convergence Overview***

Uniform (5.28), Strong (5.31), Weak (5.41), Weak-$*$ (5.46)

- Operators may be uniformly, strongly, or weakly convergent.
- Sequences may be strongly or weakly convergent. This includes sequences of elements, e.g. $(x_n)$, and sequences of operators, e.g. $(T_n)$.
- Sequences of functionals, e.g. $(\varphi_n) \subset X^*$, may be weak-$*$ convergent.

See Examples 5.33, 5.42, 5.43, and 5.50.

## 5.7 Chapter Summary

This chapter starts out by defining a *Banach space* and giving several examples (e.g. $C([a,b])$, $\ell^p(\mathbb{N})$). One of the key distinctions is finite-dimensional vs. infinite dimensional Banach spaces. We introduce *linear operators* and define what it means for a linear operator to be *bounded*. We define the *norm* of a bounded linear operator, and as an example we look at some matrix norms. The first key result is that a linear map is bounded iff it is continuous. Next we have the *Bounded Linear Transformation Theorem* which says that we can extend a linear map from a dense linear subspace of a Banach space to the entire Banach space, where the norm is preserved. Then we have the *Open Mapping Theorem*, which says that if a linear map $T$ is one-to-one and onto, then $T^{-1}$ is bounded. We define the *kernel*, *range*, *nullility*, and *rank* of a linear map, and we look at the left and right shift operators as an example.

Next we discuss the special case of finite-dimensional Banach spaces. There are 3 important differences between these and there infinite-dimensional counterparts:

1. every finite-dimensional normed linear space is a Banach space;
2. every linear operator on a finite-dimensional space is continuous;
3. all norms on a finite-dimensional space are equivalent.

Then we discuss convergence of bounded linear operators, defining *uniform*, *strong* (pointwise), and *weak* (coordinate-wise) convergence. We define what it means for a linear operator to be *compact*: $T(B)$ is precompact for every bounded set $B \Leftrightarrow$ every bounded sequence $(x_n)$ has a subsequence $(x_{n_k})$ such that the sequence $(Tx_{n_k})$ converges.

Finally, we define the (topological) *dual space*, $X^*$, as the set of continuous (equivalently, bounded) linear functionals on $X$. We show that linear functionals can be viewed as a sum of coordinate-wise maps:

$$\begin{aligned} w_i\left(\sum x_j e_j\right) &= x_i = \text{ weight of } i\text{th basis vector} \qquad (w_i \in X^*) \\ \varphi\left(\sum x_i e_i\right) &= \sum x_i \varphi(e_i) = \sum x_i \underbrace{\varphi_i}_{=\varphi(e_i)} \\ \varphi &= \sum \varphi_i w_i. \end{aligned} \tag{5.1}$$

We present the *Hahn-Banach Theorem*, which says that we can extend a bounded linear functional from a linear subspace to the entire space while preserving the norm. We define the *bidual space*, $X^{**}$. We note that each element $x \in X$ corresponds to a functional $F_x \in X^{**}$ that is defined as:

$$F_x(\varphi) = \varphi(x) \qquad \forall\, \varphi \in X^*.$$

That is, for all $\varphi \in X^*$, the functional $F_x$ evaluated at $\varphi$ is equal to the functional $\varphi$ evaluated at $x$. (Note: there may be other elements $G \in X^{**}$ that do not correspond to elements in $X$ in this way.) If every element in $F \in X^{**}$ can be identified with an element $x \in X$ as in (5.7), then we say that $X$ is *reflexive* (e.g. $L^p,\ 1 < p < \infty$). We define *weak convergence*, $x_n \rightharpoonup x$, by:

$$\varphi(x_n) \to \varphi(x) \qquad \forall\ \varphi \in X^*,$$

which corresponds to coordinate-wise convergence (see 5.1). Next, we define *weak-$*$ convergence* of bounded linear functionals, $\varphi_n \overset{*}{\rightharpoonup} \varphi$, by:

$$\varphi_n(x) \to \varphi(x) \qquad \forall\ x \in X.$$

This is a "less strong" condition than weak convergence:

$$\begin{aligned} \varphi_n \overset{*}{\rightharpoonup} \varphi \quad &\text{means} \quad \varphi_n(x) \to \varphi(x)\ \forall\ x\ \Leftrightarrow\ F_x(\varphi_n) \to F_x(\varphi)\ \forall\ F_x \\ \varphi_n \rightharpoonup \varphi \quad &\text{means} \quad F(\varphi_n) \to F(\varphi)\ \forall\ F \in X^{**}. \end{aligned}$$

We conclude the chapter with the *Banach-Alaoglu Theorem*, which says that the closed unit ball in $X^*$ is weak-$*$ compact.

# 6 Hilbert Spaces

## 6.1 Inner Products

**Definition 6.1.** ***Inner Product***
Definition 6.1 on page 125 and Notes 11/19/10

An *inner product* on a complex linear space $X$ is a map

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$$

that satisfies

(a) $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$
(b) $\langle y, x \rangle = \overline{\langle x, y \rangle}$
(c) $\langle x, x \rangle \geq 0$
(d) $\langle x, x \rangle = 0$ iff $x = 0$

Note: (a) + (b) implies that $\langle \lambda x + \mu y, z \rangle = \overline{\lambda} \langle x, z \rangle + \overline{\mu} \langle y, z \rangle$
We say that the inner product is *anti-linear* or *conjugate linear* in the first component.

**Definition 6.2.** ***Hilbert Space***
Definition 6.2 on page 126

A *Hilbert space* is an inner product space that is complete wrt the norm $\| \cdot \| = \sqrt{\langle x, x \rangle}$
Note: every Hilbert space is a Banach space.

**Example 6.3.**
Example 6.3 on page 126

The standard inner product on $\mathbb{C}^n$ is given by

$$\langle x, y \rangle = \sum_{j=1}^{n} \overline{x_j} y_j$$

This space is complete, and therefore it is a finite-dimensional Hilbert space.

**Example 6.4.**
Example 6.4 on page 126

Consider $C([a,b])$. We can define the inner product

$$\langle f,g\rangle = \int_a^b \overline{f(x)}g(x)\,dx$$

But this space is not complete, so it is not a Hilbert space. The completion of $C([a,b])$ wrt to the norm

$$\|f\| = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$$

is denoted by $L^2([a,b])$.
Note: the Banach spaces $L^p([a,b])$ are not Hilbert spaces when $p \neq 2$.

**Example 6.5.**
Example 6.5 on page 125

We can define the Hilbert space $\ell^2(\mathbb{Z})$ by

$$\ell^2(\mathbb{Z}) = \left\{ (z_n)_{n=-\infty}^{\infty} \,\middle|\, \sum_{n=-\infty}^{\infty} |z_n|^2 < \infty \right\}$$

An inner product is given by

$$\langle x,y\rangle = \sum_{n=-\infty}^{\infty} \overline{x_n} y_n$$

**Theorem 6.6.** ***Cauchy-Schwarz Inequality***
Theorem 6.8 on page 128

$$|\langle x,y\rangle| \leq \|x\|\|y\|$$

where $\|x\| = \sqrt{\langle x,x\rangle}$.

Proof Outline

- Expand $0 \leq \langle \lambda x - \mu y, \lambda x - \mu y\rangle$
- Let $\langle x,y\rangle = re^{i\theta}$. Set

$$\lambda = \|y\|e^{i\theta} \qquad \mu = \|x\|$$

**Theorem 6.7.** ***Parallelogram Law***
Theorem 6.9 on page 128

A normed linear space $X$ is an inner product space with a norm derived from its inner product iff

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall\, x, y \in X$$

Therefore, the parallelogram law holds in every Hilbert space.

Proof Outline
Inner Product $\Rightarrow$ Parallelogram Law: Follows from the properties of inner products.
Parallelogram Law $\Rightarrow$ Inner Product: An inner product can be defined by

$$\langle x, y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right\}$$

**Definition 6.8.** ***Inner Product for a Cartesian Product***
page 129

Let $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ be two inner product spaces. The inner product of the Cartesian space $X \times Y$ is

$$\langle \langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle_{X\times Y} = \langle x_1, x_2\rangle_X + \langle y_1, y_2\rangle_Y$$

The associated norm is

$$\|\,\langle x, y\rangle\,\| = \sqrt{\|x\|^2 + \|y\|^2}$$

**Theorem 6.9.**
Theorem 6.10 on page 129

An inner product is a continuous map from $X \times X \to \mathbb{C}$.

Proof Outline
Continuity is given by the Cauchy-Schwarz Inequality, which implies that

$$|\,\langle x_1, y_1\rangle - \langle x_2, y_2\rangle\,| \le \|x_1 - x_2\|\|y_1\| + \|x_2\|\|y_1 - y_2\|$$

## 6.2 Orthogonality

**Definition 6.10.** ***Orthogonal***
Definition 6.11 on page 130

- Vectors $x$ and $y$ are *orthogonal*, $x \perp y$, if $\langle x, y\rangle = 0$
- Subsets $A$ and $B$ are *orthogonal*, $A \perp B$, if $\langle a, b\rangle = 0\ \forall\, a \in A,\ b \in B$
- The *orthogonal complement* of $A$ is the set

$$A^\perp = \left\{x \in \mathcal{H} \;\middle|\; x \perp y\ \forall\, y \in A\right\}$$

**Theorem 6.11.**
Theorem 6.12 on page 130

The orthogonal complement of a subset of a Hilbert space is a closed linear subspace.

Proof Outline
**Linearity:** If $y, z \in A^\perp$, then $\lambda y + \mu z \in A^\perp$ because

$$\langle x, \lambda y + \mu z\rangle = \lambda \langle x, y\rangle + \mu \langle x, z\rangle = 0 \quad \forall\, x \in A$$

**Closed:** If $(y_n) \subset A^\perp,\ y_n \to y$, then $y \in A^\perp$ because

$$\langle x, y\rangle = \left\langle x, \lim_{n\to\infty} y_n \right\rangle = \lim_{n\to\infty} \langle x, y_n\rangle = 0$$

**Definition 6.12.** ***Orthogonal Direct Sum***
Definition 6.14 on page 133

If $\mathcal{M}$ and $\mathcal{N}$ are orthogonal closed linear subspaces of a Hilbert space, then we define the *orthogonal direct sum* by

$$\mathcal{M} \oplus \mathcal{N} = \left\{ y + z \;\middle|\; y \in \mathcal{M} \text{ and } z \in \mathcal{N} \right\}$$

**Theorem 6.13.** ***Projection***
Theorem 6.13 on page 130

Let $\mathcal{M}$ be a closed linear subspace of a Hilbert space $\mathcal{H}$.

(a) For each $x \in \mathcal{H}$ there is a unique closest point $y \in \mathcal{M}$ such that

$$\|x - y\| = \min_{z\in\mathcal{M}} \|x - z\|$$

(b) The point $y \in \mathcal{M}$ closest to $x \in \mathcal{H}$ is the unique element of $\mathcal{M}$ with the property that $(x - y) \perp \mathcal{M}$.

**Corollary:** If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. If $\mathcal{M}$ is not closed, then $\mathcal{H} = \overline{\mathcal{M}} \oplus \mathcal{M}^\perp$

Proof Outline
**Part a**

- Let $d = \inf\{\|x - z\| \;\big|\; z \in \mathcal{M}\}$
- $\exists\ (y_n) \in \mathcal{M}$ such that $\lim_{n\to\infty} \|x - y_n\| = d$
- Use the parallelogram law for $(x - y_n)$ and $(x - y_m)$ to show that $(y_n)$ is Cauchy
  Note: $(y_m + y_n)/2 \in \mathcal{M}$
- Since $\mathcal{H}$ is complete and $\mathcal{M}$ is closed, $(y_n) \to y \in \mathcal{M}$, and $\|x - y\| = d$
- Unique: assume $\|x - y\| = d$ and $\|x - y'\| = d$. Use the parallelogram law to show that $y = y'$

**Part b**

- Since $y \in \mathcal{M}$ minimizes the distance to $x$, we have $\forall\ \lambda \in \mathbb{C},\ z \in \mathcal{M}$

$$\|x-y\|^2 \le \|x-y+\lambda z\|^2$$
$$2\text{Re}\lambda \langle x-y, z\rangle \le |\lambda|^2\|z\|^2$$

- Let $\langle x-y, z\rangle = |\langle x-y, z\rangle| e^{i\varphi}$. Take $\lambda = \epsilon e^{i\varphi},\ \epsilon > 0$.

$$2|\langle x-y, z\rangle| \le \epsilon\|z\|^2$$

- Taking the limit as $\epsilon \to 0$ shows that $\langle x-y, z\rangle = 0 \Rightarrow (x-y) \perp \mathcal{M}$
- Unique: assume that $\exists\ y, y' \in \mathcal{M}$ such that $(x-y) \perp \mathcal{M}$ and $(x-y') \perp \mathcal{M}$. To show that $y = y'$, substitute $z = y - y' \in \mathcal{M}$ into

$$\left\langle z, y-y'\right\rangle = \langle z, y\rangle - \left\langle z, y'\right\rangle = 0$$

### 6.3 Orthonormal Bases

**Definition 6.14.** ***Orthonormal***
page 133

- A subset $U \in \mathcal{H}$ is *orthogonal* if any two distinct elements in $U$ are orthogonal
- A set of vectors $U$ is *orthonormal* if it is orthogonal and $\|u\| = 1\ \forall\ u \in U$
- An *orthonormal basis* of a Hilbert space is an orthonormal set such that every vector in the space can be expanded in terms of the basis

**Example 6.15.**
Example 6.17 on page 134

An orthonormal basis of $\ell^2(\mathbb{Z})$ is the set of coordinate basis vectors $\{e_n \mid n \in \mathbb{Z}\}$ given by

$$e_n = (\delta_{kn})_{k=-\infty}^{\infty}$$

For example,

$$e_{-1} = (\dots, 0, 1, 0, 0, 0, \dots), \quad e_0 = (\dots, 0, 0, 1, 0, 0, \dots), \quad e_1 = (\dots, 0, 0, 0, 1, 0, \dots)$$

**Theorem 6.16.** ***Bessel's Inequality***
Theorem 6.24 on page 137 and Notes 11/24/10

Let $U = \left\{e_\alpha \mid \alpha \in I\right\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Then

(a) $\sum_{\alpha\in I} |\langle e_\alpha, x\rangle|^2 \le \|x\|^2$ (Note: this is true in any inner product space)

(b) $\sum_{\alpha\in I} \langle e_\alpha, x\rangle e_\alpha \in \mathcal{H}$, and this series is convergent

(c) $x - \sum_{\alpha\in I} \langle e_\alpha, x\rangle e_\alpha \in U^\perp$

(d) $\|x - \sum_{\alpha\in I} \langle e_\alpha, x\rangle e_\alpha\|^2 = \|x\|^2 - \sum_{\alpha\in I} |\langle e_\alpha, x\rangle|^2$

Note: (a) is the main result, (b)-(d) can be considered corollaries.

Proof Outline

**Parts a & d**

- Compute $\|x - \sum_{\alpha \in I} \langle e_\alpha, x\rangle e_\alpha\|^2$ for any finite subset $J \subset U$:

$$\|x - \sum_{\alpha \in I} \langle e_\alpha, x\rangle e_\alpha\|^2 = \left\langle \left( x - \sum_{\alpha \in J} \langle e_\alpha, x\rangle e_\alpha \right), \left( x - \sum_{\beta \in J} \langle e_\beta, x\rangle e_\beta \right) \right\rangle$$
$$= \langle x, x\rangle - \sum_{\beta \in J} \langle e_\beta, x\rangle \langle x, e_\beta\rangle - \sum_{\alpha \in J} \overline{\langle e_\alpha, x\rangle} \langle e_\alpha, x\rangle + \sum_{\alpha,\beta \in J} \overline{\langle e_\alpha, x\rangle} \langle e_\beta, x\rangle \langle e_\alpha, e_\beta\rangle$$
$$= \|x\|^2 - \sum_{\alpha \in J} |\langle e_\alpha, x\rangle|^2$$

- For $J \subset U$ finite, this proves (d), and since the LHS is nonnegative, it also proves (a)
- For $J$ countable, take the limit as $n \to \infty$
- If $J$ is uncountable, prove that there are only finitely many nonzero terms in $\sum_{\alpha \in J} |\langle e_\alpha, x\rangle|^2$
  - Define $J_n := \left\{ \alpha \in J \;\middle|\; \langle e_\alpha, x\rangle | > \frac{1}{n} \right\}$
  - Want to prove $A_n$ is finite $\Rightarrow$ Assume $A_n$ is not finite
  - Pick $(n^2\|x\|^2 + 1)$ elements in $A_n$ and get that

$$\frac{n^2\|x\|^2 + 1}{n^2} \leq \sum_{\alpha \in J_n} |\langle e_\alpha, x\rangle|^2 \leq \|x\|^2 \;\Rightarrow \text{Contradiction}$$

  - Thus, $J_n$ is finite, which implies that $\left\{ \alpha \in J \;\middle|\; \langle e_\alpha, x\rangle \neq 0 \right\}$ is countable

**Part b**

- By the proof of (a), $\forall\, x \in \mathcal{H}$ there are at most countably many $\alpha \in J$ such that $\langle e_\alpha, x\rangle \neq 0$, so $\sum_{\alpha \in J} \langle e_\alpha, x\rangle e_\alpha = \sum_{i=1}^{\infty} \langle e_i, x\rangle e_i$
- By (a), $\sum_{i=1}^{\infty} |\langle e_i, x\rangle|^2 \leq \|x\|^2 < \infty$, so $\sum_{\alpha \in I} \langle e_\alpha, x\rangle e_\alpha$ is Cauchy
- Since $\mathcal{H}$ is complete, this series converges in $\mathcal{H}$

**Part c**

- Consider $e_{\alpha 0} \in U$

$$\left\langle \left( x - \sum_{\alpha \in I} \langle e_\alpha, x\rangle e_\alpha \right), e_{\alpha 0} \right\rangle = \langle x, e_{\alpha 0}\rangle - \sum_{\alpha \in I} \overline{\langle e_\alpha, x\rangle} \langle e_\alpha, e_{\alpha 0}\rangle$$
$$= \langle x, e_{\alpha 0}\rangle - \langle x, e_{\alpha 0}\rangle = 0$$

**Definition 6.17.** ***Complete and Maximal Orthonormal Set***
page 133 and Notes 11/29/10

An orthonormal set $U \subset \mathcal{H}$ is *complete* if $U^\perp = \{0\}$. Equivalently, $U$ is a *maximal orthonormal set* in $\mathcal{H}$ if for every $V \in \mathcal{H}$ such that $U$ is a strict subset of $V$, $V$ is not orthonormal. A complete orthonormal set is an orthonormal basis.

**Theorem 6.18.** ***Parseval's Identity and more...***
Theorems 6.26 & 6.28 on page 139 and Notes 11/29/10

If $U = \{e_\alpha \mid \alpha \in A\}$ is an orthonormal subset of $\mathcal{H}$, TFAE:

(a) $U$ is an orthonormal basis for $\mathcal{H}$, i.e. $\forall\, x \in \mathcal{H},\; x = \sum_{\alpha \in A} \langle e_\alpha, x\rangle\, e_\alpha$

(b) $U$ is complete $\Leftrightarrow$ $U$ is a maximal orthonormal set

(c) Parseval's Identity: $\|x\|^2 = \sum_{\alpha \in A} |\langle x, e_\alpha\rangle|^2$

Proof Outline

**(a) $\Rightarrow$ (b)**

- $\forall\, x \in U^\perp \in \mathcal{H},\; x = \sum_{\alpha \in A} \langle e_\alpha, x\rangle\, e_\alpha = 0$

**(b) $\Rightarrow$ (c)**

- $\forall\, x \in \mathcal{H}$, we have from Bessel's Inequality (c) and (d) (Theorem 6.16 in pdf) that:

$$x - \sum_{\alpha \in A} \langle e_\alpha, x\rangle\, e_\alpha \in U^\perp = \{0\}$$

$$\|x\|^2 - \sum_{\alpha \in A} |\langle e_\alpha, x\rangle|^2 = \left\| x - \sum_{\alpha \in A} \langle e_\alpha, x\rangle\, e_\alpha \right\|^2 = 0$$

**(c) $\Rightarrow$ (a)**

- Combining Perseval's Identity and Bessel's Inequality (c) (Theorem 6.16 in pdf) we have that $\forall\, x \in \mathcal{H}$:

$$\left\| x - \sum_{\alpha \in A} \langle e_\alpha, x\rangle\, e_\alpha \right\|^2 = \|x\|^2 - \sum_{\alpha \in A} |\langle e_\alpha, x\rangle|^2 = 0$$

- This implies that $x = \sum_{\alpha \in A} \langle e_\alpha, x\rangle\, e_\alpha$

**Theorem 6.19.** ***Parseval's Identity, General Version***
Theorem 6.28 on page 139

Let $U = \{e_\alpha \mid \alpha \in A\}$ be an orthonormal basis of $\mathcal{H}$. Then $\forall\, x, y \in \mathcal{H},\; x = \sum_{\alpha \in A} a_\alpha e_\alpha$ and $y = \sum_{\alpha \in A} b_\alpha e_\alpha$, and

$$\langle x, y\rangle = \sum_{\alpha \in A} \overline{a_\alpha} b_\alpha$$

**Theorem 6.20.**
Theorem 6.29 on page 140 and Notes 11/29/10

Every Hilbert space has an orthonormal basis. If $U$ is an orthonormal set, then $\mathcal{H}$ has an orthonormal basis containing $U$.

Proof Outline

- Introduce a partial order $\leq$ by set inclusion: $U \leq V$ if $U \subset V$
- If $\{U_\alpha \mid \alpha \in A\}$ is a totally ordered family of orthonormal sets, then $\bigcup_{\alpha \in A} U_\alpha$ is an orthonormal set and is an upper bound (in the sense of inclusion) of the family $\{U_\alpha \mid \alpha \in A\}$
- Zorn's Lemma implies that $\bigcup_{\alpha \in A} U_\alpha$ has a maximal element, and by Theorem 6.18 this element is an orthonormal basis

**Definition 6.21.** ***Gram-Schmidt***
page 140 and Notes 11/29/10

The *Gram-Schmidt* algorithm enables us to construct an orthonormal basis from any countable linearly independent set whose linear span is dense in $\mathcal{H}$.
Given $\{x_1, x_2, \dots\}$ linearly independent vectors, construct and orthonormal set such that $\text{span}\,\{x_n\}_{n=1}^N = \text{span}\,\{e_n\}_{n=1}^N$

$$\begin{aligned}
\text{Let } y_1 &= x_1 \;\Rightarrow\; e_1 = \frac{y_1}{\|y_1\|} \\
y_2 &= x_2 - \langle e_1, x_2 \rangle\, e_1 \;\Rightarrow\; e_2 = \frac{y_2}{\|y_2\|} \\
y_n &= x_n - \sum_{i=1}^{n-1} \langle e_i, x_n \rangle\, e_i \;\Rightarrow\; e_n = \frac{y_n}{\|y_n\|}
\end{aligned}$$

**Definition 6.22.** ***Separable***
page 133 and Notes 11/29/10

A Hilbert space is separable if it contains a countable dense subset.
Equivalently, a Hilbert space is separable if it has a finite or countable basis. (See Theorem 6.24)

**Definition 6.23.** ***Isomorphic***
page 133 and Notes 11/29/10

Let $(X_1, \langle \cdot, \cdot \rangle_1)$ and $(X_2, \langle \cdot, \cdot \rangle_2)$ be two inner product spaces. They are *isomorphic* if there exists an isomorphism $T : X_1 \to X_2$ such that $\langle Tx, Ty \rangle_2 = \langle x, y \rangle_1 \;\forall\, x, y \in X_1$.

Two Hilbert spaces whose orthonormal bases have the same cardinality are isomorphic, as any linear map that identifies basis elements is an isomorphism.

**Theorem 6.24.**
Notes 11/29/10

A Hilbert space is separable iff it contains a countable (orthonormal) basis.
Let $U$ be a countable basis of a separable Hilbert space, $\mathcal{H}$, and let $N = \text{card}(U)$.
If $N < \infty$, then $\mathcal{H}$ is isomorphic to $\mathbb{C}^N$.
If $N = \infty$, then $\mathcal{H}$ is isomorphic to $\ell^2(\mathbb{N})$.

Proof Outline
**Separable $\Rightarrow$ Countable Orthonormal Basis**

- Suppose span $\{x_n\}_{n=1}^{\infty}$ is a countable dense subset of $\mathcal{H}$
- span $\{x_n\}_{n=1}^{\infty}$ has a maximal linearly independent subset, $\{y_n\}_{n=1}^{N}$, with $1 \le N < \infty$ or $N = \infty$
- Applying Gram-Schmidt to $\{y_n\}_{n=1}^{N}$ yields a countable orthonormal basis

**Countable Orthonormal Basis $\Rightarrow$ Separable**

- If $\{e_n\}_{n=1}^{N}$ is an orthonormal basis, then the set $\left\{x = \sum_{n=1}^{N} a_n e_n \;\middle|\; \mathrm{Re}(a_n),\ \mathrm{Im}(a_n) \in \mathbb{Q}\right\}$ is a dense countable subset of $\mathcal{H}$, so $\mathcal{H}$ is separable
- To construct the isomorphisms, define:

$$T : \mathcal{H} \to \begin{cases} \mathbb{C} & N < \infty \\ \ell^2(\mathbb{N}) & N = \infty \end{cases}$$
$$x \to \{\langle e_n, x\rangle\}_{n=1}^{N}$$

- By Parseval's Identity (6.18 in pdf), $\|x\|^2 = \sum_{n=1}^{N} |\langle e_n, x\rangle|^2$, so if $N = \infty$ then $Tx \in \ell^2(\mathbb{N})$
- $T$ is one-to-one and onto
- $T$ preserves the inner product, and is therefore an isomorphism, since

$$\begin{aligned}\langle x, y\rangle &= \left\langle \sum_{i=1}^{N} \langle e_i, x\rangle\, e_i, \sum_{i=1}^{N} \langle e_i, y\rangle\, e_i \right\rangle \\ &= \overline{\langle e_i, x\rangle}\, \langle e_i, y\rangle \\ &= \langle Tx, Ty\rangle_{\ell^2 \text{ or } \mathbb{C}}\end{aligned}$$

## 6.4 Chapter Summary

This chapter begins by defining an *inner product* and a *Hilbert space*, using $\mathbb{C}^n$, $L^2([a,b])$, $\ell^2(\mathbb{Z})$, $\mathbb{C}^{m\times n}$, and $H^k([a,b])$ as examples. We establish the *Cauchy-Schwarz Inequality* and the *Parallelogram Law*. We prove that inner products are continuous maps.

Using the inner product, we define what it means for vectors and subsets to be *orthogonal*, and we also define the *orthogonal complement* of a subset. We prove the *Projection Theorem*, which says that if we have a closed linear subspace $\mathcal{M}$, then for all $x \in \mathcal{H}$ there is a unique element $y \in \mathcal{M}$ such that $y$ is the closest element in $\mathcal{M}$ to $x$ and $x - y \perp \mathcal{M}$. Using the Projection Theorem, we prove that for a closed subspace $\mathcal{M}$ we can decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, and for any subspace $\mathcal{M}$ we can decompose $\mathcal{H}$ as $\mathcal{H} = \overline{\mathcal{M}} \oplus \mathcal{M}^{\perp}$.

The last topic in this chapter is that of *orthonormal bases*. Every Hilbert space has an orthonormal basis. A key distinction is that between *separable* Hilbert spaces (those with a finite or countably infinite orthonormal basis) and nonseparable Hilbert spaces. We define what it means for an unordered sum to *converge absolutely* and to *converge unconditionally* (all sums in this chapter are interpreted to be unordered sums). For an orthonormal subset $\{u_\alpha\}$, *Bessel's Inequality* tells us that $\sum |\langle u_\alpha, x\rangle|^2 \le \|x\|^2$. For an orthonormal basis $\{u_\alpha\}$, *Parseval's Identity* tells us that $\langle x, y\rangle = \sum_{\alpha\in I} \overline{a_\alpha} b_\alpha$, where $a_\alpha = \langle u_\alpha, x\rangle$, $b_\alpha = \langle u_\alpha, y\rangle$, $x = \sum_{\alpha\in I} a_\alpha u_\alpha$, $y = \sum_{\alpha\in I} b_\alpha u_\alpha$.

**Note:** When expanding a vector $x$ in terms of an orthonormal basis $\{u_\alpha\}$, we put the $x$ in the linear (second) argument. This ensures that for any orthonormal set $U = \{u_\alpha \mid \alpha \in J\}$, $x - \sum_{\alpha \in J} \langle u_\alpha, x \rangle u_\alpha \perp U$. Otherwise, we would have:

$$\begin{aligned}
\left\langle x - \sum_{\alpha \in J} \langle x, u_\alpha \rangle u_\alpha, u_\beta \right\rangle &= \langle x, u_\beta \rangle - \left\langle \sum_{\alpha \in J} \langle x, u_\alpha \rangle u_\alpha, u_\beta \right\rangle \\
&= \langle x, u_\beta \rangle - \sum_{\alpha \in J} \overline{\langle x, u_\alpha \rangle} \langle u_\alpha, u_\beta \rangle \\
&= \langle x, u_\beta \rangle - \overline{\langle x, u_\beta \rangle} \neq 0.
\end{aligned}$$

# A Appendix

## A.1 Definitions

### A.2 Theorems