Document: Math 201B (Winter 2011)
Professor: Hunter
Latest Update: June 4, 2013
Author: Jeff Irion
http://www.math.ucdavis.edu/~jlirion

Contents

0 Measure Theory

0.1 Key Theorems

Theorem 0.1. ***Fubini's Theorem***
`http://en.wikipedia.org/wiki/Fubini%27s_theorem`

Suppose A and B are complete measure spaces. Suppose $f(x,y)$ is $A \times B$ measurable. If

$$\int_{A\times B} |f(x,y)|\, d(x,y) < \infty$$

where the integral is taken with respect to a product measure on the space over $A \times B$, then

$$\int_A \left(\int_B f(x,y)\, dy \right) dx = \int_B \left(\int_A f(x,y)\, dx \right) dy = \int_{A\times B} f(x,y)\, d(x,y)$$

the first two integrals being iterated integrals with respect to two measures, respectively, and the third being an integral with respect to a product of these two measures.

Corollary:
If $f(x,y) = g(x)h(y)$ for some functions g and h, then

$$\int_A g(x)\, dx \int_B h(y)\, dy = \int_{A\times B} f(x,y)\, d(x,y)$$

the third integral being with respect to a product measure.

Theorem 0.2. ***Tonelli's Theorem***
`http://en.wikipedia.org/wiki/Fubini%27s_theorem#Tonelli.27s_theorem`

Suppose that A and B are σ-finite measure spaces, not necessarily complete. If either

$$\int_A \left(\int_B |f(x,y)|\, dy \right) dx < \infty \text{ or } \int_B \left(\int_A |f(x,y)|\, dx \right) dy < \infty$$

then

$$\int_{A\times B} |f(x,y)|\, d(x,y) < \infty$$

and

$$\int_A \left(\int_B f(x,y)\, dy \right) dx = \int_B \left(\int_A f(x,y)\, dx \right) dy = \int_{A\times B} f(x,y)\, d(x,y)$$

Remark 0.3. ***Fubini vs. Tonelli***
http://en.wikipedia.org/wiki/Fubini%27s_theorem

Tonelli's theorem is a successor of Fubini's theorem. The conclusion of Tonelli's theorem is identical to that of Fubini's theorem, but the assumptions are different. Tonelli's theorem states that on the product of two -finite measure spaces, a product measure integral can be evaluated by way of an iterated integral for nonnegative measurable functions, regardless of whether they have finite integral. A formal statement of Tonelli's theorem is identical to that of Fubini's theorem, except that the requirements are now that (X, A, μ) and (Y, B, ν) are σ-finite measure spaces, while f maps $X \times Y$ to $[0, \infty]$.

Theorem 0.4. ***Hölder's Inequality***
Theorem 12.54 on page 356

Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, then $fg \in L^1(X, \mu)$ and

$$\left| \int fg \, d\mu \right| \leq \|f\|_p \|g\|_q$$

Theorem 0.5. ***Young's Inequality***
Theorem 12.58 on page 359

Let $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

Theorem 0.6. ***Lebesgue Dominated Convergence Theorem***
Theorem 12.35 on page 348

Suppose that (f_n) is a sequence of integrable functions, $f_n : X \to \overline{\mathbb{R}}$, on a measure space $(X, \mathcal{A}, \mu)$ that converges pointwise to a limiting function $f : X \to \overline{\mathbb{R}}$. If there is an integrable function $g : X \to [0, \infty]$ such that

$$|f_n(x)| \leq g(x) \quad \forall \, x \in X, \; n \in \mathbb{N}$$

then f is integrable and

$$\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$$

Theorem 0.7. ***Cauchy-Schwarz Inequality***
http://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality

Formal Statement: For all vectors x, y of an inner product space,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$$
$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Square of a Sum:

$$\left|\sum_{i=1}^{n} x_i y_i\right|^2 \leq \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2$$

In L^2:

$$\left|\int f(x)g(x)\,dx\right|^2 \leq \int |f(x)|^2\,dx \int |g(x)|^2\,dx$$

7 Fourier Series

7.1 Fourier Series

Definition 7.1. ***2π-periodic***
page 149

A function $f : \mathbb{R} \to \mathbb{C}$ is 2π-*periodic* if

$$f(x + 2\pi) = f(x) \quad \forall\, x \in \mathbb{R}$$

A 2π-periodic function may be indentified with a function on the unit circle, or one-dimensional torus, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The space $C(\mathbb{T})$ is the space of continuous functions from $\mathbb{T}$ to $\mathbb{C}$, and $L^2(\mathbb{T})$ is the completion of $C(\mathbb{T})$ with respect to the L^2-norm,

$$\|f\| = \left(\int_{\mathbb{T}} |f(x)|^2 \, dx \right)^{1/2}$$

$L^2(\mathbb{T})$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{T}} \overline{f(x)} g(x) \, dx$$

Definition 7.2. $L^p(\mathbb{T})$
page 92 and Notes 1/3/11

$L^p(\mathbb{T}) :=$ the space of Lebesgue measurable functions, $f : \mathbb{T} \to \mathbb{C}$ such that $\int_{\mathbb{T}} |f|^p \, dx < \infty$, where $1 \le p < \infty$. We define the L^p-norm as:

$$\|f\|_p = \left(\int_{\mathbb{T}} |f|^p \, dx \right)^{1/p}$$

For $p = \infty$, $L^\infty(\mathbb{T})$ is the space of Lebesgue measurable functions that are *essentially bounded* on $\mathbb{T}$, meaning that f is bounded on every subset of $\mathbb{T}$ with nonzero measure. The norm on $L^\infty(\mathbb{T})$ is the *essential supremum*

$$\|f\|_\infty = \inf\{M \;\big|\; |f(x)| \le M \text{ a.e. in } \mathbb{T}\}$$

We identify f with g if $f = g$ a.e. (almost everywhere, except possibly on a set of measure 0).

Theorem 7.3.
Notes 1/3/11

$L^p(\mathbb{T})$ with the norm $\|f\|_{L^p} = (\int_{\mathbb{T}} |f|^p \, dx)^{1/p}$ is a Banach space.

Theorem 7.4.
Notes 1/3/11

$C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ for $1 \le p < \infty$.

Note: $C(\mathbb{T}) :=$ the space of continuous functions $f : \mathbb{T} \to \mathbb{C}$

Proposition 7.5.
Notes 1/3/11

$p > q \;\Rightarrow\; L^p(\mathbb{T}) \subset L^q(\mathbb{T}) \quad \text{and} \quad \|\cdot\|_p \ge \|\cdot\|_q$

Also,
$L^1(\mathbb{T}) \supset L^2(\mathbb{T}) \supset \ldots \supset C(\mathbb{T})$

Example 7.6. ***Fourier Basis Example***
Notes 1/3/11

$$\sum_{n\neq 0} \frac{1}{|n|} e^{inx} = f(x)$$

$$\sum_{n\neq 0} \frac{1}{|n|^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$\lim_{N\to\infty} \int \left| f(x) - \sum_{n=-N,\ n\neq 0}^{N} \frac{1}{|n|} e^{inx} \right|^2 dx = 0$$

Line 2 and Bessel's Inequality tell us that the series converges in $L^2(\mathbb{T})$. However, it doesn't converge pointwise everywhere on $\mathbb{T}$.

Ex. at $x = 0$, $\sum_{n\neq 0} \frac{1}{|n|}$ diverges.

Proposition 7.7. ***Orthonormal Basis of $L^2(\mathbb{T})$***
page 150

The Fourier basis elements are the functions

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

$\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{T})$.

Proof Outline

- Orthogonality

It is easily shown that

$$\langle e_m, e_n \rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

- Completeness
  This proof relies upon the ideas of convolution and approximate identities. (See Theorems 7.12 and 7.13.)

Definition 7.8. ***Convolution***
page 150

The *convolution* of two continuous functions $f, g : \mathbb{T} \to \mathbb{C}$ is the continuous function $f * g : \mathbb{T} \to \mathbb{C}$ defined by the integral

$$(f * g)(x) = \int_{\mathbb{T}} f(x - y) g(y)\, dy$$

Using the change of variable $z = x - y$, it is seen that

$$(f * g)(x) = \int_{\mathbb{T}} f(z) g(x - z)\, dz$$

so that $f * g = g * f$.

Definition 7.9. ***Approximate Identity***
Definition 7.1 on page 151

A family of functions $\{\varphi_n \in C(\mathbb{T}) \mid n \in \mathbb{N}\}$ is an *approximate identity* if

(a) $\varphi_n(x) \geq 0$

(b) $\displaystyle\int_{\mathbb{T}} \varphi_n(x)\, dx = 1$

(c) $\displaystyle\lim_{n \to \infty} \int_{\delta \leq |x| \leq \pi} \varphi_n(x)\, dx = 0 \quad \forall\, \delta > 0$

Note: in (c), $\mathbb{T}$ is identified with $[-\pi, \pi]$.

Theorem 7.10.
Theorem 7.2 on page 151 and Notes 1/5/11 and FA 49

If $\{\varphi_n \in C(\mathbb{T}) \mid n \in \mathbb{N}\}$ is an approximate identity and $f \in C(\mathbb{T})$, then $\varphi_n * f$ converges uniformly to f as $n \to \infty$.

Note: the term "approximate identity" comes from this result, since $\{\varphi_n\}$ is an *approximation to the identity*.

Proof

$$f(x) = \int_{\mathbb{T}} \varphi_n(y) f(x)\, dy$$
$$(\varphi_n * f)(x) = \int_{\mathbb{T}} \varphi_n(y) f(x-y)\, dy$$
$$(\varphi_n * f)(x) - f(x) = \int_{\mathbb{T}} \varphi_n(y)[f(x-y) - f(x)]\, dy$$

- f is uniformly continuous, so there exists M such that $|f(x)| \leq M \ \forall\ x \in \mathbb{T}$
- $\exists\ \delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $|x - y| < \delta$

$$\begin{aligned}
|(\varphi_n * f)(x) - f(x)| &\leq \int_{-\pi}^{\pi} \varphi_n(y)|f(x-y) - f(x)|\, dy \\
&\leq \int_{|y|<\delta} \varphi_n(y)|f(x-y) - f(x)|\, dy + \int_{|y|\geq\delta} \varphi_n(y)|f(x-y) - f(x)|\, dy \\
&\leq \epsilon \int_{|y|<\delta} \varphi_n(y)\, dy + \int_{|y|\geq\delta} \varphi_n(y)[|f(x-y)| + |f(x)|]\, dy \\
&\leq \epsilon + 2M \int_{|y|\geq\delta} \varphi_n(y)\, dy
\end{aligned}$$

Using property (c) of an approximate identity gives that $\varphi_n * f \to f$ uniformly in $C(\mathbb{T})$.

Remark 7.11. ***Revised Approximate Identity Definition***
Notes 1/5/11

More generally, $\varphi_n \in L^1(\mathbb{T})$ is an approximate identity if

$$\text{(a)} \quad \int_{\mathbb{T}} |\varphi_n(x)|\, dx \leq M \quad \forall\ n \in \mathbb{N}$$
$$\text{(b)} \quad \int_{\mathbb{T}} \varphi_n(x)\, dx = 1$$
$$\text{(c)} \quad \lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} \varphi_n(x)\, dx = 0 \quad \forall\ \delta > 0$$

Theorem 7.12. ***Weierstrass Approximation Theorem***
Theorem 7.3 on page 152 and Notes 1/5/11

The trigonometric polynomials are dense in $C(\mathbb{T})$ with respect to the uniform norm.

<u>Proof</u>

- Let $f \in C(\mathbb{T})$
- Generate an approximate identity that is a trigonometric polynomial

– Define $\varphi_n = c_n(1+\cos x)^n = c_n[2\cos^2(\frac{x}{2})]^n$ and choose c_n such that $\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$

– To show φ_n is an approximate identity, we need to show that $\forall\ \delta > 0,\ \lim_{n\to\infty} \int_{|x|>\delta} \varphi_n(x)\,dx = 0$

  * Fix $\epsilon > 0$. $\forall\ x,\ \delta \le |x| \le \pi,\ \exists\ r \in (0,1)$ such that

$$\begin{aligned}(1+\cos x) &< r(1+\cos y)\\ \varphi_n(x) &< r^n \varphi_n(y)\\ \delta\varphi_n(x) &< r^n \int_{-\delta/2}^{\delta/2} \varphi_n(y)\,dy\\ \delta\varphi_n(x) &< r^n\end{aligned}$$

$$0 \le \varphi_n(x) < \frac{r^n}{\delta} \quad \forall\ x \text{ such that } \delta \le |x| \le \pi$$

– So $\varphi_n \to 0$ uniformly on $\delta \le |x| \le \pi$ as $n \to \infty$, and $\int_{|x|>\delta} \varphi_n(x)\,dx \to 0$ as $n \to \infty$

- φ_n is an approximate identity, so $\varphi_n * f$ is a trigonometric polynomial, and $\varphi_n * f$ converges uniformly to f (See Theorem 7.10)

Corollary 7.13.
page 153 and 155 and Notes 1/5/11

The trigonometric polynomials are dense in $L^2(\mathbb{T})$. That is, for any $f \in L^2(\mathbb{T})$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$$

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx}\,dx$$

If $f \in L^2(\mathbb{T})$ then the Fourier series of f converges pointwise to f a.e. (Carleson).

Proof
Let $f \in L^2(\mathbb{T})$.

- Choose $g \in C(\mathbb{T})$ such that $\|f-g\|_{L^2} < \epsilon/2$. We can do this because $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.
- Pick a trigonometric polynomial p such that $\|g-p\|_{L^2} < \epsilon/2\sqrt{2\pi}$.
- $\|g-p\|_{L^2} = (\int |g-p|^2 dx)^{1/2} \le \|g-p\|_\infty \sqrt{2\pi}$
- $\|f-p\|_{L^2} \le \|f-g\|_{L^2} + \|g-p\|_{L^2} < \epsilon/2 + \epsilon/2$

Corollary 7.14.
Notes 1/5/11

$\{\frac{1}{\sqrt{2\pi}} e^{inx} \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{T})$.

Definition 7.15. ***Periodic Fourier Transform***
page 153 and Notes 1/7/11

The *Periodic Fourier Transform* $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ maps a function to its sequence of Fourier coefficients by

$$\mathcal{F}f = \left(\hat{f}_n\right)_{n=-\infty}^{\infty}$$

Thus, the L^2 norm of a function can be computed by

$$\int_{\mathbb{T}} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} \left|\hat{f}_n\right|^2$$

This implies that $(\hat{f}_n) \in \ell^2(\mathbb{Z})$. Furthermore, the Projection Theorem (6.13 in the book) implies that

$$f_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

is the best approximation of f by a trigonometric polynomial of degree N in the L^2-norm.

Theorem 7.16. ***Parseval's Theorem***
Notes 1/7/11

Given $f, g \in L^2(\mathbb{T})$, then

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} a_n e^{inx} \\ g(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} b_n e^{inx} \\ \langle f, g\rangle &= \sum_{n\in\mathbb{Z}} \overline{a_n} b_n \end{aligned}$$

Proposition 7.17.
Proposition 7.4 on page 154

If $f, g \in L^2(\mathbb{T})$, then $f * g \in C(\mathbb{T})$ and

$$\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$$

Proof

$$(f * g)(x) = \int_{\mathbb{T}} f(x-y) g(y) \, dy$$

If $f, g \in C(\mathbb{T})$, then we can apply the Cauchy-Schwarz Inequality to get

$$|f * g(x)| \leq \|f\|_{L^2} \|g\|_{L^2}$$

Taking the supremum of both sides yields

$$\|f * g\|_\infty \leq \|f\|_{L^2} \|g\|_{L^2}$$

If $f, g \in L^2(\mathbb{T})$, then there exist sequences $(f_k), (g_k) \in C(\mathbb{T})$ such that $\|f - f_k\|_2 \to 0$ and $\|g - g_k\|_2 \to 0$ as $k \to \infty$. Also, the sequence $(f_k * g_k) \in C(\mathbb{T})$ is Cauchy with respect to the sup-norm, since

$$\begin{aligned} \|f_j * g_j - f_k * g_k\| &\le \|(f_j - f_k) * g_j\|_\infty + \|f_k * (g_j - g_k)\|_\infty \\ &\le \|f_j - f_k\|_2 \|g_j\|_2 + \|f_k\|_2 \|g_j - g_k\|_2 \\ &\le M\left(\|f_j - f_k\|_2 \; + \|g_j - g_k\|_2\right) \end{aligned}$$

where $M \ge \|f_j\|_2$ and $M \ge \|g_k\|_2$, since the sequences converge in $L^2(\mathbb{T})$. Since $C(\mathbb{T})$ is complete, the sequence $(f_k * g_k)$ converges uniformly to a continuous function $f * g \in C(\mathbb{T})$, and $f * g$ satisfies the inequality.

Theorem 7.18. ***Convolution Theorem***
Theorem 7.5 on page 154 and Notes 1/10/11

If $f, g \in L^2(\mathbb{T})$, then

$$\text{(Book)} \qquad \widehat{(f * g)}_n = \sqrt{2\pi}\,\hat{f}_n \hat{g}_n$$

$$\text{(Notes)} \qquad \widehat{(f * g)}_n = \hat{f}_n \hat{g}_n$$

Proof Outline
Compute $\widehat{(f * g)}_n$, using Fubini's Theorem to change the order of integration.

Remark 7.19. ***Alternative bases for L^2***
page 155 and Notes 1/7/11

The non-normalized orthogonal basis:

$$\{e^{inx}\}$$

$$\hat{f}_n = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx}\, dx$$

Sines and Cosines:

$$\{1, \cos(nx), \sin(nx) \mid n = 1, 2, 3, \ldots\}$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)$$

$$a_0 = \frac{1}{\pi} \int_{\mathbb{T}} f(x)\, dx \qquad a_n = \frac{1}{\pi} \int_{\mathbb{T}} f(x) \cos(nx)\, dx \qquad b_n = \frac{1}{\pi} \int_{\mathbb{T}} f(x) \sin(nx)\, dx$$

7.2 L^1 Functions

Remark 7.20. ***L^1 Functions***
Notes 1/7/11

$L^1(\mathbb{T})$ is the space of periodic functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_{L^1} = \int_{\mathbb{T}} |f(x)|\, dx < \infty$$

Note that $L^1(\mathbb{T})$ is a Banach space but not a Hilbert space. We can define the Fourier coefficients of f as

$$c_n = \int_{\mathbb{T}} f(x) e^{-inx}\, dx$$

Note that $|c_n| \leq \int |f(x)|\, dx$. We can write the trigonometric polynomial approximation of f as

$$f(x) \sim \sum_{n \in \mathbb{Z}} c_n e^{inx}$$

However, this does not necessarily converge to f.

Lemma 7.21. ***Riemann-Lebesgue Lemma***
Notes 1/7/11 and 1/10/11

If $f \in L^1(\mathbb{T})$ has Fourier coefficients c_n, then $c_n \to 0$ as $|n| \to \infty$.

Proof Outline (1/7/11)

- Prove for smooth functions (use Integration By Parts)
- Approximate non-smooth functions with smooth functions

Proof Outline (1/10/11)

- Fix $\epsilon > 0$
- The trigonometric polynomials are dense in $L^1(\mathbb{T})$, so we can pick a trigonometric polynomial p such that $\|f - p\|_{L^1} < \epsilon$
- If $\deg p = N$ and $n > N$, then

$$\begin{aligned} |\hat{f}(n)| &= \frac{1}{2\pi} \left| \int f e^{-inx}\, dx \right| \\ &= \frac{1}{2\pi} \left| \int (f - p) e^{-inx}\, dx \right| \qquad \text{Note: } \int p e^{-inx}\, dx = 0 \ \forall\ n > N \text{ by orthogonality} \\ &\leq \frac{1}{2\pi} \|f - p\|_{L^1} \\ &\leq \frac{\epsilon}{2\pi} < \epsilon \end{aligned}$$

Definition 7.22. ***Fourier Transform for $L^1(\mathbb{T})$***
Notes 1/10/11

The *Fourier Transform* $\mathcal{F} : f \to \hat{f}$, $\mathcal{F} : L^1(\mathbb{T}) \to C_0(\mathbb{Z})$

$$C_0(\mathbb{Z}) = \left\{ (c_n)_{n\in\mathbb{Z}} \;\middle|\; c_n \to 0 \text{ as } |n| \to \infty \right\}$$
$$\|(c_n)\|_\infty = \max_{n\in\mathbb{Z}} |c_n|$$

$\mathcal{F}$ is a bounded linear map, with $\|\mathcal{F}f\|_\infty \le \|f\|_{L^1}$
Note: $\mathcal{F}$ is not onto.

Example 7.23. ***$\mathcal{F}$ is not onto***
Notes 1/10/11

There is no function whose Fourier coefficients are

$$\hat{f}(n) = \frac{i \ \mathrm{sgn}(n)}{\log |n|} \qquad |n| \ge 2$$

7.3 Kernels and Summability Methods

Definition 7.24. ***Dirichlet Kernel***
Notes 1/10/11 and FA 44

The *Dirichlet kernel* is

$$D_N(x) = \frac{1}{2\pi} \sum_{|n|\le N} e^{inx} = \frac{1}{2\pi} \left[\frac{\sin\left((N+\frac{1}{2})x\right)}{\sin(\frac{x}{2})} \right] \qquad x \ne 0$$
$$D_N(0) = \frac{1}{2\pi}(2N+1)$$

(See the Kernel Overview.)

Derivation of the Dirichlet Kernel

Suppose $f \in L^1(\mathbb{T})$, $f(x) \sim \sum \hat{f}_n e^{inx}$. Define the N^{th} partial sum of the Fourier series of f as

$$\begin{aligned} S_N(f)(x) &= \sum_{|n|\le N} \hat{f}_n e^{inx} \\ &= \frac{1}{2\pi} \sum_{|n|\le N} \left(\int f(y) e^{-iny}\, dy \right) e^{inx} \\ &= \frac{1}{2\pi} \int \left(\sum_{|n|\le N} e^{in(x-y)} \right) f(y)\, dy \\ &= \int D_N(x-y) f(y)\, dy = D_N * f \end{aligned}$$

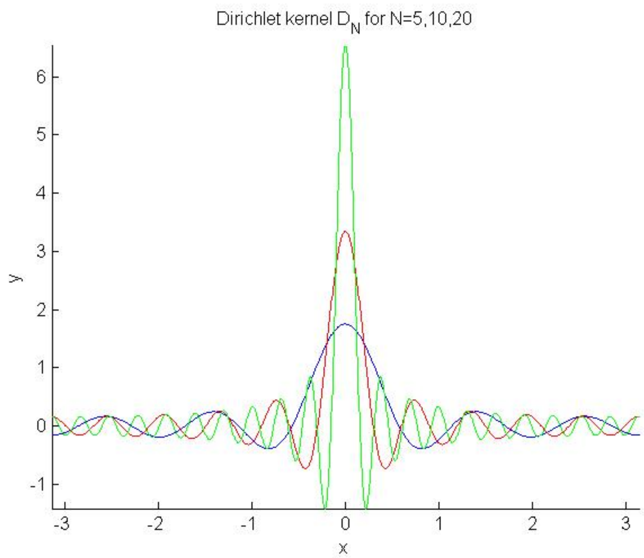

Figure 1: Dirichlet kernels.

Example 7.25. ***D_N is not an approximate identity***
Notes 1/12/11

The Dirichlet kernel is not an approximate identity.

(a) $$\int D_N\, dx = \int \left(\frac{1}{2\pi} \sum e^{inx} \right) dx = \frac{1}{2\pi} \cdot 2\pi = 1$$

(b) $$\int \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \le |D_N|\, dx \le \frac{4}{\pi^2} \left(\sum_{k=1}^{N} \frac{1}{k} \right) + 2 + \frac{\pi}{4}$$

As $N \to \infty$, $\displaystyle\int |D_N|\, dx = \frac{4}{\pi} \log N + O(1) \to \infty$ as $N \to \infty$

(c) For $\delta > 0$, $\displaystyle\lim_{N\to\infty} \int_{|x|>\delta} |D_N|\, dx \not\to 0$

Thus, we can't conclude that if $f \in C(\mathbb{T})$ or $f \in L^1(\mathbb{T})$ then $D_N * f \to f$ uniformly

Theorem 7.26. ***Absolute Convergence***
HW 3 Problem 2 and FA page 41

If $f \in C(\mathbb{T})$ and its Fourier series is absolutely convergent, $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, then the Fourier series converges uniformly to f.

Let $\mathcal{A}(\mathbb{T})$ denote the space of integrable functions whose Fourier coefficients are absolutely convergent. That is, $f \in \mathcal{F}(\mathbb{T})$ if $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$. If $f \in \mathcal{A}(\mathbb{T})$, then $f \in C(\mathbb{T})$.

Definition 7.27. ***Summability Method: Cesáro Summation***
Notes 1/12/11 and FA 52

The N^{th} Cesáro sum of a series is the average of the first N partial sums in the series:

$$\sigma_N = \frac{s_0 + s_1 + \ldots + s_{N-1}}{N}$$

Example 7.28. ***Cesáro Summation Example***
Notes 1/12/11

Consider the series $\sum_{n=1} (-1)^n = 1 - 1 + 1 - 1 + 1 \ldots$. Then the nth partial sum is

$$S_N = \begin{cases} 1 & N \text{ odd} \\ 0 & N \text{ even} \end{cases}$$

Consider the averages of partial sums:

$$\sigma_N = \frac{S_1 + \ldots + S_N}{N}$$
$$\sigma_N = \begin{cases} \frac{1}{2} & N \text{ even} \\ \frac{\frac{1}{2}(N+1)}{N} = \frac{1}{2} + \frac{1}{2N} & N \text{ even} \end{cases} \to \frac{1}{2} \text{ as } N \to \infty$$

Thus, $\sum_{n=1} (-1)^n = \frac{1}{2}$ (C).

Theorem 7.29.
Notes 1/14/11

Cesáro summation is *regular*, meaning that if $\sum a_n = s$ then $\sum a_n = s$ (C).

Definition 7.30. ***Fejér Kernel***
Notes 1/12/11

The Fejér Kernel is:

$$K_N(x) = \frac{1}{2\pi} \sum_{|n| \le N} \left(1 - \frac{|n|}{N+1}\right) e^{inx}$$

$$K_N(x) = \frac{1}{2\pi(N+1)} \left[\frac{\sin\left(\frac{(N+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right]^2$$

(See the Kernel Overview.)

Proof (that the two forms are equivalent)

- Consider
$$\left[\frac{1}{2}\left(e^{ix} + e^{-ix}\right) - 1\right] K_N(x) = \frac{1}{2\pi N}\left(\frac{1}{2}e^{i(N+1)x} + \frac{1}{2}e^{-i(N+1)x} - 1\right)$$
- Use the fact that
$$\left(\sin\frac{x}{2}\right)^2 = -\frac{1}{4}\left(e^{ix} - 2 + e^{-ix}\right)$$

Derivation of the Fejér Kernel
Form the N^{th} Cesáro mean of the Fourier series:

$$\begin{aligned}\sigma_N(f)(x) &= \frac{S_0 f + S_1 f + \ldots + S_N f}{N+1} \\ &= \frac{1}{2\pi} \sum_{|n| \le N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n) e^{inx} \\ &= K_N * f\end{aligned}$$

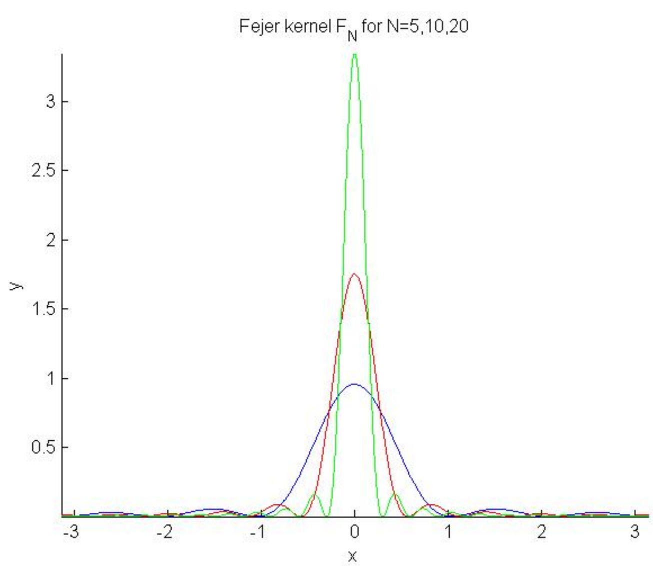

Figure 2: Fejér kernels.

Theorem 7.31.
Notes 1/12/11

K_N is an approximate identity. If $f \in C(\mathbb{T})$, then $\sigma_N f = K_N * f \to f$ uniformly and if $f \in L^p(\mathbb{T})$, then $\sigma_N f = K_N * f \to f$ in $L^p(\mathbb{T})$.

Corollary 7.32.
1/12/11

Suppose $f, g \in L^1(\mathbb{T})$ and $\hat{f} = \hat{g}$. Then $f = g$.

Proof

- Set $h = f - g$
- Then $\hat{h}(n) = 0$
- $K_N * h \to h$ in L^1
- $K_N * h = 0\ \forall\ N$, so $h = 0 \Rightarrow f = g$

Note: we could have used the original approximate identity for this proof.

Definition 7.33. ***Summability Method: Abel Summation***
Notes 1/14/11

$$S = \sum_{n=0}^{\infty} a_n$$

$$S = \lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^n \quad \text{(A)}$$

Theorem 7.34.
Notes 1/14/11

Abel summation is regular.

Proof

- We will use summation by parts. Suppose $S = \sum_{n=0}^{\infty} a_n$, $S_n = \sum_{k=0}^{n} a_k$, $S_n \to S$ as $n \to \infty$

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n r^n &= a_0 + \sum_{n=1}^{\infty}(S_n - S_{n-1})r^n && \text{(Since } a_n = S_n - S_{n-1}\text{)} \\
&= a_0 + \sum_{n=1}^{\infty}(S_n - S_n r^{n+1}) && \text{(re-index)} \\
&= a_0 + (1-r)\sum_{n=1}^{\infty}(S_n r^n) - S_0 r \\
&= (1-r)\sum_{n=0}^{\infty} S_n r^n && (S_0 = a_0)
\end{aligned}$$

$$\left|\sum_{n=0}^{\infty}(a_n r^n) - s\right| = (1-r)\left|\sum_{n=0}^{\infty}(S_n - S)r^n\right| \le (1-r)\sum_{n=0}^{\infty}|S_n - S|r^n \qquad 1 = (1-r)\sum_{n=0}^{\infty} r^n$$

$$S = (1-r)\sum_{n=0}^{\infty} S r^n$$

- Fix $\epsilon > 0$. Choose N such that $|S_n - S| < \epsilon/2$ for $n > N$. Then

$$\left|\sum_{n=0}^{\infty} a_n r^n - S\right| < (1-r)\sum_{n=0}^{N}|S_n - S|r^n + \frac{\epsilon}{2}\underbrace{(1-r)\sum_{n=N+1}^{\infty} r^n}_{\le 1}$$

- Choose $(1-r) < \delta$, where $\delta \sum_{n=0}^{N} |S_n - S| < \epsilon/2$
- $n > N \Rightarrow \left|\sum_{n=0}^{\infty} a_n r^n - S\right| < \epsilon/2 + \epsilon/2 = \epsilon$

Theorem 7.35. ***Tauber & Littlwood***
Notes 1/14/11

Suppose that $\lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^n$ exists and $n a_n = O(1)$ as $n \to \infty$. (i.e. there is an M such that $|n a_n| \le M \ \forall\ n$.) Then $\sum a_n$ exists (and is equal to the limit).

Definition 7.36. ***Poisson Kernel***
Notes 1/14/11

Identify $\mathbb{T}$ as the unit circle in $\mathbb{C}$, i.e.

$$\mathbb{T} = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\} \Leftrightarrow z = e^{i\theta}$$

$$\begin{aligned} f(\theta) &\sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{in\theta} \\ \hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta \\ f_r(\theta) &= \sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} e^{in\theta} \\ &= P_r * f(\theta) \end{aligned}$$

The Poisson kernel is

$$\begin{aligned} P_r(\theta) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}, \quad 0 < r < 1 \\ P_r(\theta) &= \frac{1}{2\pi}\left[\frac{1-r^2}{1-2r\cos\theta + r^2}\right] \\ P_r(0) &= \frac{1}{2\pi}\frac{1-r^2}{(1-r)^2} \end{aligned}$$

(See the Kernel Overview.)

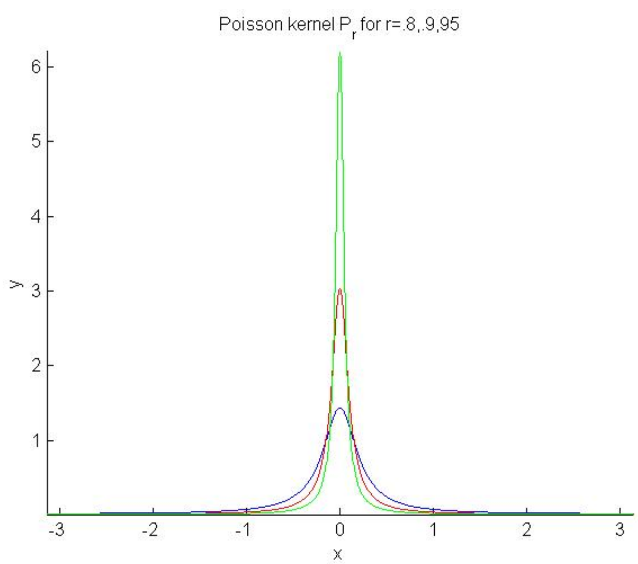

Figure 3: Poisson kernels.

Remark 7.37. ***Properties of the Poisson Kernel***
Notes 1/14/11

- The Poisson kernel is not a trigonometric polynomial
- The Poisson kernel satisfies:
  (a) $\int P_r(\theta)\, d\theta = 1$
  (b) $P_r \geq 0$
  (c) $P_r(\theta) \to 0$ uniformly as $r \to 1^-$ on $\delta < |\theta| < \pi$

Theorem 7.38.
Notes 1/14/11

P_r is an approximate identity as $r \to 1^-$.

Corollary 7.39.
Notes 1/14/11

If $f \in L^p(\mathbb{T}), 1 \leq p < \infty$, then $P_r * f \to f$ as $r \to 1^-$.
If $f \in C(\mathbb{T})$, then $P_r * f \to f$ uniformly.

Remark 7.40. ***Kernel Overview***

Dirichlet

- Equations:
  - $D_N(x) = \frac{1}{2\pi} \sum_{|n| \leq N} e^{inx}$
  - $D_N(x) = \frac{1}{2\pi} \left[\frac{\sin\left((N+\frac{1}{2})x\right)}{\sin(\frac{x}{2})} \right], \quad x \neq 0$
  - $D_N(0) = \frac{1}{2\pi}(2N+1)$
- Summability Method: Standard
- Approximate Identity: No

Fejér

- Equations:
  - $K_N(x) = \frac{1}{2\pi} \sum_{|n| \leq N} \left(1 - \frac{|n|}{N+1}\right) e^{inx}$
  - $K_N(x) = \frac{1}{2\pi(N+1)} \left[\frac{\sin\left(\frac{(N+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \right]^2$
- Summability Method: Cesáro
- Approximate Identity: Yes

Poisson

- Equations:
  - $P_r(\theta) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta}, \quad 0 < r < 1$
  - $P_r(\theta) = \frac{1}{2\pi} \left[\frac{1-r^2}{1-2r\cos\theta + r^2} \right]$
  - $P_r(0) = \frac{1}{2\pi} \frac{1-r^2}{(1-r)^2}$
- Summability Method: Abel
- Approximate Identity: Yes, as $r \to 1^-$

7.4 Harmonic Functions

Definition 7.41. ***Harmonic***
Notes 1/19/11

Let $\Omega \subset \mathbb{R}^n$ be an open set.
$u : \Omega \to \mathbb{R}$ is *harmonic* on Ω if $\Delta u = 0$ in Ω.

Recall: $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$

Remark 7.42. ***Harmonic & Analytic Functions***
Notes 1/19/11

There is a close connection in 2-D between harmonic and analytic (holomorphic) functions.

$$F : \Omega \to \mathbb{C}$$
$$F(z) = u(x, y) + iv(x, y)$$

where u, v satisfy the Cauchy-Riemann equations:

$$\left.\begin{array}{r} u_x = v_y \\ u_y = -v_x \end{array}\right\} \Rightarrow u_{xx} + v_{yy} = 0$$

Example 7.43. $\Delta u = 0$ ***on the Complex Unit Disk***
Notes 1/19/11

Consider the Dirichlet problem on $D = \{(x, y) \subset \mathbb{R}^2 \mid x^2 + y^2 < 1\}$:

$$\Delta u = 0 \text{ in } D$$
$$u = f \text{ on } \partial D = \pi$$

Here $f \in C(\partial D)$.
Want $u \in C^2(D) \cap C(\overline{D})$.
Use separation of variables:

$$u(r, \theta) = F(r)G(\theta)$$

We get that:

$$G(\theta) = e^{in\theta}$$
$$F(r) = Ar^n + Br^{-n} \quad n \neq 0$$
$$F(r) = A + B \ln r \quad n = 0$$

We want the solution to belong to $C^2(D)$, so we set

$$F(r) = r^{|n|}, \quad n \in \mathbb{Z}$$
$$\Rightarrow u(r, \theta) = \sum_{n \in \mathbb{Z}} c_n r^{|n|} e^{in\theta}$$

We want that:

$$u(1, \theta) = f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$$
$$\Rightarrow c_n = \hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} \, d\theta$$

Note that:

$$u(r, \theta) = \underbrace{(P_r * f)(\theta)}_{\text{Green's function}} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta}$$

Remark 7.44.
Notes 1/19/11

$P_r(\theta)$ is a $C^\infty(D)$ function of r, θ in $0 \le r < 1$, and

$$\Delta P_r(\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 P_r}{\partial \theta^2} = 0$$

Theorem 7.45.
Notes 1/19/11

Suppose that $f \in C(\partial D)$. Then $u(r, \theta) = (P_r * f)(\theta)$ is a solution of

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

Moreover, $u \in C^\infty(D) \cap C(\overline{D})$.

Proof

- $u(r, \theta) = \int_{\mathbb{T}} P_r(\theta - \phi) f(\phi)\, d\phi$ (by Lebesgue Dominated Convergence Theorem)
- So $u \in C^\infty(D)$, and $\Delta u = 0$
- Moreover, $P_r * f \to f$ uniformly as $r \to 1^-$
- So $u \in C(\overline{D})$

Theorem 7.46.
Notes 1/19/11

There is a unique solution $u \in C^2(D) \cap C(\overline{D})$ of the Dirichlet problem. (Can be proved using the maximum principle and/or energy estimates.)

Corollary 7.47.
Notes 1/19/11

Every harmonic function $u \in C^2(D) \cap C(\overline{D})$ is smooth and has the mean value property:

$$u(r = 0) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

7.5 Hausdorff-Young Inequality

Remark 7.48. ***Background Info/Review***
Notes 1/21/11

- Function Spaces
  - Let $1 \leq p < \infty$. If $f \in L^p(\mathbb{T})$, then $f : \mathbb{T} \to \mathbb{C}$ and $\|f\|_p = \left(\int_{\mathbb{T}} |f|^p \, dx\right)^{1/p} < \infty$.
  - $f = g$ in L^p if $f = g$ a.e.
  - In L^∞, $\|f\|_\infty = \operatorname{ess\,sup}_{\mathbb{T}} |f(x)| = \inf_{\text{measure } N=0} \sup\{|f(x)| \mid x \in \mathbb{T} \setminus N\}$
- Sequence Spaces
  - Let $1 \leq q < \infty$. If $\hat{f} \in \ell^q(\mathbb{Z})$, $\hat{f} : \mathbb{Z} \to \mathbb{C}$, then $\|\hat{f}\|_q = \left(\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^q\right)^{1/q} < \infty$
  - In ℓ^∞, $\|\hat{f}\|_\infty = \sup_{n \in \mathbb{Z}} |\hat{f}(n)|$
- Question: When is $\mathcal{F} : L^p(\mathbb{T}) \to \ell^q(\mathbb{Z})$, $f \mapsto \hat{f}$, a bounded linear map?
  - $\mathcal{F} : L^2 \to \ell^2$
    - $\|\mathcal{F}f\|_{\ell^2} = \frac{1}{\sqrt{2\pi}} \|f\|_{L^2}$
    - $\mathcal{F}$ is onto
  - $\mathcal{F} : L^1 \to C_0 \subset \ell^\infty$
    - $\|\mathcal{F}f\|_{\ell^\infty} \leq \frac{1}{2\pi} \|f\|_{L^1}$

Theorem 7.49. ***Hausdorff-Young Theorem/Inequality***
Notes 1/21/11

Suppose $1 \leq p \leq 2$ and $2 \leq p' \leq \infty$ are Hölder conjugates ($\frac{1}{p} + \frac{1}{p'} = 1$).
Then $\mathcal{F} : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ is a bounded linear map, i.e. $\|\hat{f}\|_{\ell^{p'}} \leq C_p \|f\|_{L^p}$.

Remark 7.50.
Notes 1/21/11

1. Interpolation result (Riesz-Thorin Theorem)
2. $\mathcal{F}$ is *not* onto if $1 \leq p < 2$.
   - Ex: $p = 1$, $p' = \infty$, then $f \in L^1 \to \hat{f} \in C_0 \Rightarrow$ not all of ℓ^∞
   - $\sum\limits_{|n| \geq 2} \frac{i \text{ sgn}(n)}{\log n} e^{inx}$ is not the Fourier series of any L^1 function
3. This result does not hold for $2 < p \leq \infty$
4. If $f \in L^p$ (or even if $f \in C$), one can't say much about the Fourier coefficients $\hat{f}$ beyond the fact that $f \in L^p$ so $\hat{f} \in \ell^2$
   - Example:

$$f(x) = \sum_{n=2}^{\infty} \frac{e^{in \log n}}{n^{1/2} (\log n)^2} e^{inx}$$
$$\hat{f}(n) = \frac{e^{in \log n}}{n^{1/2} (\log n)^2}$$
$$\sum |\hat{f}(n)|^2 = \sum \frac{1}{n (\log n)^4} < \infty$$

$\hat{f} \in \ell^2$ so $f \in L^2$. Is $\hat{f} \in \ell^p$ for $p < 2$, e.g. $p = 2 - \epsilon$?

$$\sum |\hat{f}(n)|^{2-\epsilon} = \sum \frac{1}{n^{1-\epsilon/2} (\log n)^{4-2\epsilon}} = \infty$$

So $\hat{f} \notin \ell^{p'}$ for any $p' < 2$

7.6 Fourier Series of Differentiable Functions (Section 7.2 in H&N)

Definition 7.51. ***Fourier Series Differentiation***
Notes 1/24/11

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$$
$$f'(x) = \sum_{n \in \mathbb{Z}} inc_n e^{inx}$$
$$\mathcal{F} : \frac{d}{dx} \mapsto in$$

Proposition 7.52.
Notes 1/24/11

If $f \in C^1(\mathbb{T})$, then
$$\widehat{f'}(n) = in\hat{f}(n)$$
(Actually, it is sufficient that $f \in L^1(\mathbb{T})$.)

See Definition 7.56 and Proposition 11.21.

Definition 7.53. ***Orders***
Notes 1/24/11

If $\phi, \psi : \mathbb{Z} \to \mathbb{C}$, we say that

- $\phi = O(\psi)$ as $|n| \to \infty$ if there exists C such that $|\phi(n)| \leq C|\psi(n)|\ \forall\ n \in \mathbb{Z}$
- $\phi = o(\psi)$ as $|n| \to \infty$ if $\lim\limits_{|n|\to\infty} \left|\frac{\phi(n)}{\psi(n)}\right| = 0$

Theorem 7.54.
Notes 1/24/11

If $f \in C^1(\mathbb{T})$, then $\hat{f}(n) = o(\frac{1}{n})$ as $|n| \to \infty$

If $f \in C^k(\mathbb{T})$, where $k \in \mathbb{N}$, then $\hat{f}(n) = o(\frac{1}{n^k})$ as $|n| \to \infty$

Proof

- $\widehat{f'}(n) = in\hat{f}(n)$ if $f \in C^1$
- $\hat{f}(n) = \frac{1}{in}\widehat{f'}(n),\ n \neq 0$, and $\widehat{f'}(n) \to 0$ as $|n| \to \infty$ by the Riemann-Lebesgue Lemma
- So $\hat{f}(n) = o(\frac{1}{n})$ as $|n| \to \infty$
- In general, $\hat{f}(n) = \frac{1}{(in)^k}\widehat{f^k}(n) = o(\frac{1}{n^k})$

Corollary 7.55.
page 157 and Notes 1/24/11

If $f \in C^\infty(\mathbb{T})$, then $\lim\limits_{|n|\to\infty} |n|^k\hat{f}(n) = 0\ \forall\ k \in \mathbb{N}$.

In other words, the Fourier coefficients of smooth functions form a rapidly decreasing sequence that decreases faster than any polynomial. Heuristically, a smooth function contains a small amount of high frequency components.

Compare to Theorem 11.18.

Definition 7.56. ***Weak L^2-derivatives (1)***
Notes 1/24/11

Suppose that $f \in L^2(\mathbb{T})$ such that $\sum_{n\in\mathbb{Z}} n^2|\hat{f}(n)|^2 < \infty$. Then we define the *weak L^2-derivative* $g = f' \in L^2(\mathbb{T})$ by

$$g(x) = \sum_{n\in\mathbb{Z}} in\hat{f}(n)e^{inx}$$

See Proposition 7.52 and Proposition 11.21.

Definition 7.57. ***Sobolev Space (1)***
page 158 and Notes 1/24/11

$$H^1(\mathbb{T}) = \{f \in L^2(\mathbb{T}) \mid f' \in L^2(\mathbb{T})\}$$

$$\langle f, g\rangle_{H^1} = \int_{\mathbb{T}} (\overline{f}g + \overline{f'}g')\,dx = \sum_{n\in\mathbb{Z}} (1+n^2)\overline{\hat{f}(n)}g(n)$$

$$\|f\|_{H^1} = \left[\int_{\mathbb{T}} (|f|^2 + |f'|^2)\,dx\right]^{1/2}$$

In other words, $f \in H^1(\mathbb{T})$ iff f and its weak derivative f' (defined by integration by parts) belong to $L^2(\mathbb{T})$.

Definition 7.58. ***Integration By Parts***
Notes 1/24/11

For $f, g \in H^1$:

$$\begin{aligned}
\int_{\mathbb{T}} \overline{f'}g\,dx &= 2\pi \sum \overline{\widehat{f'}(n)}\hat{g}(n) \\
&= 2\pi \sum \overline{in\hat{f}(n)}\hat{g}(n) \\
&= -2\pi \sum \overline{\hat{f}(n)}in\hat{g}(n) \\
&= -2\pi \sum \overline{\hat{f}(n)}\widehat{g'}(n) \\
&= -\int_{\mathbb{T}} \overline{f}g'\,dx
\end{aligned}$$

Definition 7.59. ***Weak Derivative (2)***
page 159 and Notes 1/24/11

A function $g \in L^1(\mathbb{T})$ is the *weak derivative* of a function $f \in L^1(\mathbb{T})$, written $g = f'$, if for every $\phi \in C^\infty(\mathbb{T})$ we have

$$\int_{\mathbb{T}} f\phi'\,dx = -\int_{\mathbb{T}} g\phi\,dx$$

In other words, we are using integration by parts $(\int_{\mathbb{T}} \overline{f'} g\,dx = -\int_{\mathbb{T}} \overline{f} g'\,dx)$, to define f' pointwise a.e. We determine $\hat{g}(n)\ \forall\ n$ by choosing $\phi = e^{-inx}$.

Compare to Distributional Derivative, Definition 11.10.

Example 7.60. ***Weak Derivative of*** $f(x) = |x|$
Notes 1/26/11

$$f(x) = |x| \qquad -\pi < x < \pi$$

$f \in C(\mathbb{T})$, but its standard derivative $f' \notin C(\mathbb{T})$ because $f'(0)$ and $f'(\pi)$ don't exist. We shall see if $g = f'$ (weak derivative) exists. We want:

$$\begin{aligned}
\int g\phi\,dx &= -\int f\phi'\,dx \\
&= -\int_0^{\pi} x\phi'\,dx + \int_{-\pi}^{0} x\phi'\,dx \\
&= -x\phi\Big|_0^{\pi} + \int_0^{\pi} \phi\,dx + x\phi\Big|_{-\pi}^{0} - \int_{-\pi}^{0} \phi\,dx \\
&= \cancel{-\pi\phi(\pi)} + \cancel{\pi\phi(-\pi)} + \int_{-\pi}^{\pi} \operatorname{sgn} x\,\phi\,dx
\end{aligned}$$

We conclude that $\int f\phi\, x = -\int g\phi\,dx\ \forall\ \phi \in C^\infty(\mathbb{T})$ if $g(x) = \operatorname{sgn} x$.

Example 7.61. ***Weak Derivative of*** $f(x) = \operatorname{sgn} x$
Notes 1/26/11

$$
\begin{aligned}
\int h\phi\,dx &= -\int g\phi'\,dx \\
&= -\int_0^{\pi} \phi'\,dx + \int_{-\pi}^{0} \phi'\,dx \\
&= -\left[\phi(\pi) - \phi(0)\right] + \left[\phi(0) - \phi(-\pi)\right] \\
&= 2\left[\phi(0) - \phi(\pi)\right]
\end{aligned}
$$

There is no such $h \in L^1$. To see this, take $\phi = \frac{1}{2\pi} e^{-inx} \in C^\infty(\mathbb{T})$.

$$\hat{h}(n) = \frac{1}{\pi}[1 - e^{in\pi}] = \begin{cases} \frac{2}{\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

This contradicts the Riemann-Lebesgue Lemma, and therefore there is no such $h \in L^1$.

Proposition 7.62.
Notes 1/26/11

f is weakly differentiable with $f \in L^1$ iff it is absolutely continuous.

Definition 7.63. ***Absolutely Continuous***
`http://en.wikipedia.org/wiki/Absolute_continuity#Absolute_continuity_of_functions`

f is *absolutely continuous* if it has a derivative f' a.e., the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

Theorem 7.64.
Notes 1/26/11

If f is weakly differentiable with weak derivative $g = f' \in L^1(\mathbb{T})$, then

$$\hat{g}(n) = in\hat{f}(n)$$

<u>Proof</u>

$$\hat{g}(n) = \frac{1}{2\pi}\int g(x)e^{-inx}\,dx = -\frac{1}{2\pi}\int f(x)e^{-inx}\,dx = in\hat{f}(n)$$

Proposition 7.65.
Notes 1/26/11

A function $f \in L^2(\mathbb{T})$ has a weak derivative $g \in L^2(\mathbb{T})$ iff

$$\sum_{n\in\mathbb{Z}} n^2 |\hat{f}(n)|^2 < \infty$$

and then

$$g(x) = \sum_{n\in\mathbb{Z}} in\hat{f}(n)e^{inx}$$

Definition 7.66. ***Sobolev Space (2)***
Notes 1/26/11

The Sobolev space $W^{1,p}(\mathbb{T})$, $1 \le p \le \infty$, consists of all functions $f : \mathbb{T} \to \mathbb{C}$ s.t. $f \in L^p(\mathbb{T})$, $f' \in L^p(\mathbb{T})$. If $p = 2$, we write $W^{1,2}(\mathbb{T}) = H^1(\mathbb{T})$ (where the H is because it is a Hilbert space).

A function $f \in H^1(\mathbb{T})$ iff

$$\sum_{n\in\mathbb{Z}} (1+n^2)|\hat{f}(n)|^2 < \infty$$

and

$$\begin{aligned}\|f\|_{H^1} &= \left(\int |f|^2\,dx + \int |f'|^2\,dx\right)^{1/2} \\ &= \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2\right)^{1/2} \\ &= \left(2\pi \sum_{n\in\mathbb{Z}} (1+n^2)|\hat{f}(n)|^2\right)^{1/2}\end{aligned}$$

Theorem 7.67. ***Sobolev Embedding Theorem***
Notes 1/26/11

If $f \in H^1(\mathbb{T})$ then $f \in C(\mathbb{T})$ and

$$\|f\|_\infty \le C\|f\|_{H^1}$$

$J : H^1 \to C$ (Embedding), $f \mapsto f$.

Proof

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\hat{f}(n)| &= \sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^{1/2}}(1+n^2)^{1/2}|\hat{f}(n)| \\ &\le \left(\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^{1/2}}\right)\left(\sum_{n\in\mathbb{Z}} (1+n^2)|\hat{f}(n)|\right) \\ &\le C\|f\|_{H^1}\end{aligned}$$

It follows that $f \in C(\mathbb{T})$ because the Fourier series converges uniformly to f (see Theorem 7.26) and

$$\|f\|_\infty \leq \sum_{n\in\mathbb{Z}} |\hat{f}(n)| \leq C\|f\|_{H^1}$$

7.7 Chapter Summary

This chapter explores the spaces $L^p(\mathbb{T})$, $p \in [1,\infty)$, with special attention given to the Hilbert space $L^2(\mathbb{T})$. These spaces are the completion of $C(\mathbb{T})$ with respect to the L^p-norm; thus, $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ for $p \in [1,\infty)$. Since $\mathbb{T}$ has finite Lebesgue measure, we can use Hölder's Inequality to show that for $p > q$, $\|\cdot\|_p \geq \|\cdot\|_q$, which implies that $L^p(\mathbb{T}) \subset L^q(\mathbb{T})$. We define the *convolution* of two functions and what it means for a family of functions to be an *approximate identity*, and we use these tools to prove the *Weierstrass Approximation Theorem*, which says that the trigonometric polynomials are dense in $C(\mathbb{T})$ with respect to the uniform norm. Since uniform convergence implies L^2 convergence, it follows that the functions $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ form an orthonormal basis for $L^2(\mathbb{T})$. Thus, for all $f \in L^2(\mathbb{T})$, we have that

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx},$$

where the equality is in the L^2 sense. A result from Carleson tells us that the Fourier series of f converges pointwise to f a.e.

Next we explore some properties of Fourier series and Fourier coefficients. Let $f, g \in L^2(\mathbb{T})$. We use the density of $C(\mathbb{T})$ in $L^2(\mathbb{T})$ to prove the *Convolution Theorem*, which allows us to express the Fourier coefficients of $f * g$ in terms of those of f and g: $\widehat{(f * g)}_n = \sqrt{2\pi}\hat{f}_n\hat{g}_n$. *Parseval's Theorem* allows us to compute $\langle f, g\rangle$ using the Fourier coefficients of f and g: $\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \overline{\hat{f}_n}\hat{g}_n$.

Now we examine the Fourier series of differentiable functions. Using integration by parts, we show that

$$\hat{f}'_n = in\hat{f}_n.$$

This gives us the concept of a *weak derivative*, since the derivative of f may not be continuous; e.g. $f(x) = |x|$. We define the *Sobolev space* $H^k(\mathbb{T})$ as the space of $L^2(\mathbb{T})$ functions with k weak derivatives. And since the boundary terms on $\mathbb{T}$ vanish, we have that $\langle f', g\rangle = -\langle f, g'\rangle$ for $f, g \in H^1(\mathbb{T})$. Thus, we may define the weak derivative of a function using integration by parts: $g \in L^1(\mathbb{T})$ is the weak derivative of $f \in L^1(\mathbb{T})$ if

$$\int_{\mathbb{T}} f\phi'\,dx = -\int_{\mathbb{T}} g\phi\,dx \qquad \forall\ \phi \in C^\infty(\mathbb{T}).$$

Finally, we prove a special case of the *Sobolev Embedding Theorem*: if $f \in H^k(\mathbb{T})$ for $k > 1/2$, then $f \in C(\mathbb{T})$.

In addition, Hunter briefly discussed $L^1(\mathbb{T})$. We can define the Fourier series of an L^1 function, but we cannot guarantee that it converges to the function. Our main result is the *Riemann-Lebesgue Lemma*, which says that the Fourier coefficients of an L^1 function decay to zero as $n \to \infty$. Hunter then discussed 3 kernels: the *Dirichlet kernel* (standard summation), *Fejér kernel* (*Cesáro summation*), and *Poisson kernel* (*Abel summation*). These kernels are related to the concept of approximate identities, and we convolve the kernels with a function f. He covered harmonic functions, and our main result is that we can use the Poisson kernel to solve the two-dimensional Laplace equation.

11 Distributions and the Fourier Transform

11.1 Periodic Distributions

Definition 11.1. ***Test Functions***
Notes 1/28/11 and http://en.wikipedia.org/wiki/Distribution_%28mathematics%29 and Hunter's Notes page 51

We define our space of *test functions* as:

$$\mathcal{D}(\mathbb{T}) = C^{\infty}(\mathbb{T}) \text{ with the following topology:}$$

$\varphi_n \to \varphi \in \mathcal{D}$ if $\varphi_n^{(k)} \to \varphi^{(k)}$ uniformly for all $k = 0, 1, 2, \ldots$. Note that this topology is not obtained from any norm, but rather it is derived.

Definition 11.2. ***Distribution***
Notes 1/28/11 and Hunter's Notes page 51

A *distribution* is a continuous linear functional, T, that maps a set of test functions, $\mathcal{D}(\mathbb{T})$, onto the set of complex numbers. The space of distributions is denoted by $\mathcal{D}'(\mathbb{T})$. For $T \in \mathcal{D}'(\mathbb{T})$, $\varphi \in \mathcal{D}(\mathbb{T})$, we write:

$$\langle T, \varphi \rangle = T(\varphi)$$

$\mathcal{D}'(\mathbb{T})$ is the topological dual space of the distributions on $\mathbb{T}$ (i.e. $\mathcal{D}(\mathbb{T})$), with the topology defined as follows: $T_n \rightharpoonup T$ in $\mathcal{D}'$ if $\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle$ in $\mathbb{C}$ $\forall\ \varphi \in \mathcal{D}$.

$$T : \mathcal{D}(\mathbb{T}) \to \mathbb{C}$$
$$\text{Linear: } \langle T, \lambda\varphi + \mu\psi \rangle = \lambda \langle T, \varphi \rangle + \mu \langle T, \psi \rangle$$
$$\text{Continuous: If } \varphi_n \to \varphi \in \mathcal{D}, \text{ then } \langle T, \varphi_n \rangle \to \langle T, \varphi \rangle \in \mathbb{C}$$

Compare Distributional Convergence, $T_n \rightharpoonup T$ in $\mathcal{D}'$ if $\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle$, to Weak Convergence (Definition 8.41): $x_n \rightharpoonup x$ if $\langle x_n, y \rangle \to \langle x, y \rangle \quad \forall\ y \in \mathcal{H}$.

Definition 11.3. ***Seminorm***
Notes 1/28/11

Our topology on $\mathcal{D}$ is obtained from a countable family of *seminorms*:

$$\|\varphi\|_k = \sup_{x \in \mathbb{T}} |\varphi^{(k)}(x)|, \qquad k = 0, 1, 2, \ldots$$

A seminorm has the same properties as a norm except that it may assign length zero to nonzero vectors.

Example 11.4. ***Seminorms***
Notes 1/28/11

$$d(\varphi, \psi) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|\varphi - \psi\|_k}{1 + \|\varphi - \psi\|_k}$$

- This is not a norm because you can't pull out a constant
- This turns $\mathcal{D}$ into a *Fréchet space* (a complete, metrizable topological vector space topology defined by a countable family of seminorms)
- We could instead use norms to define the topology on $\mathcal{D}(\mathbb{T})$:

$$\|\varphi\|_{C^k} = \sum_{j=0}^{k} \|\varphi\|_j$$

Remark 11.5.
Notes 1/28/11

Note that the differentiation operator

$$D : \mathcal{D}(\mathbb{T}) \to \mathcal{D}(\mathbb{T}), \quad D(\varphi) = \varphi'$$

is continuous: if $\varphi_n \to \varphi \in \mathcal{D}$, then $D\varphi_n \to D\varphi \in \mathcal{D}$. This is because there are inifinitely many semi-norms.

Example 11.6. ***Regular Distribution***
page 292 and Notes 1/28/11

If $f : \mathbb{T} \to \mathbb{C}$ is integrable, $f \in L^1(\mathbb{T})$, define

$$T_f : \mathcal{D}(\mathbb{T}) \to \mathbb{C}$$
$$T_f(\varphi) = \int_{\mathbb{T}} f\varphi \, dx$$

$|T_f(\varphi)| \le \sup |\varphi| \cdot \int |f| \, dx < \infty$, so T_f is well-defined. It is a distribution because it satisfies:

1. **Linearity: (1)** $T_f(\varphi + \psi) = \int f(\varphi + \psi) \, dx = T_f(\varphi) + T_f(\psi)$. **(2)** $T_f(c\varphi) = cT_f(\varphi)$
2. **Continuity:** If $\varphi_n \to 0$ in $\mathcal{D}$, then $|T_f(\varphi_n)| \le \sup |\varphi_n| \|f\|_{L^1} \to 0$ as $n \to \infty$.
   So $T_f(\varphi_n) \to 0$ and T_f is continuous.

We identify f with T_f. Thus, $L^1(\mathbb{T}) \subset D^1(\mathbb{T})$.

We call T_f a *regular distribution*. A regular distribution is a distribution that is given by the integration of a test function φ against a function f.

Definition 11.7. ***Principal Value Distribution***
page 293

A *principal value distribution* is a singular distribution, denoted by p.v. $(1/x)$, and its action on a test function φ is given by

$$\text{p.v. } \frac{1}{x}(\varphi) = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} \frac{\varphi(x)}{x}\, dx$$

Example 11.8.
Notes 1/28/11

Consider the periodic δ-function (actually a distribution, not a function).

$$\begin{aligned}
&\langle \delta, \varphi \rangle = \varphi(0) \\
&\langle \delta, \varphi + \psi \rangle = (\varphi + \psi)(0) = \varphi(0) + \psi(0) = \langle \delta, \varphi \rangle + \langle \delta, \psi \rangle \\
&\langle \delta, c\varphi \rangle = c\,\langle \delta, \varphi \rangle
\end{aligned}$$

$\varphi_n \to 0$ implies $\varphi_n(0) \to 0$, and therefore δ is a continuous linear functional.

δ is not regular. Proof:

- Suppose $\langle \delta, \varphi \rangle = \int f\varphi\, dx$ for some $f \in L^1$.
- Consider $\varphi_n(x) = \left[\frac{1+\cos x}{2}\right]^n$
- $\langle \delta, \varphi_n \rangle = 1\ \forall\, n$, but $\int f\varphi_n\, dx \to 0$ as $n \to \infty$ by the Lebesge-Dominated Convergence Theorem if $f \in L^1$
- Thus, there is no function $f \in L^1$ such that $\int f\varphi\, dx = \varphi(0)$

Example 11.9.
Notes 1/28/11

$$\text{Let } T_n = \begin{cases} \frac{1}{2}n & |x| \le \frac{1}{n} \\ 0 & \frac{1}{n} \le |x| \le \pi \end{cases}$$

Then $\int_{-\pi}^{\pi} T_n\, dx = 1\ \forall\, n$. Claim: $\langle T_n, \varphi \rangle = \frac{n}{2}\int_{1/n}^{1/n} \varphi(x) \to \varphi(0)$ as $n \to \infty$. Proof:

$$\begin{aligned}
\left| \frac{n}{2} \int_{-1/n}^{1/n} \varphi(x)\, dx - \varphi(0) \right| &= \frac{n}{2} \left| \int_{-1/n}^{1/n} [\varphi(x) - \varphi(0)]\, dx \right| \\
&\le \frac{n}{2} \left[\sup_{|x| \le 1/n} |\varphi(x) - \varphi(0)| \right] \cdot \frac{2}{n} \\
&\le \sup_{|x| \le 1/n} |\varphi(x) - \varphi(0)| \to 0 \text{ as } n \to \infty
\end{aligned}$$

Definition 11.10. ***Distributional Derivative***
page 295

Every distribution $T \in \mathcal{D}'(\mathbb{T})$ has a distributional derivative $T' \in \mathcal{D}(\mathbb{T})$ that is given by

$$\left\langle T', \phi \right\rangle = -\left\langle T, \phi' \right\rangle \qquad \forall\ \phi \in \mathcal{D}(\mathbb{T})$$

Compare to Weak Derivative (2), Definition 7.59.

Definition 11.11. ***Motivation for Distributional Derivatives***
Notes 1/31/11

Suppose $f \in C^\infty$ is a smooth function. Consider $T_{f'}$:

$$\left\langle T_{f'}, \varphi \right\rangle = \int f' \varphi \, dx = -\int f \varphi' \, dx = -\left\langle T_f, \varphi' \right\rangle$$

Want: $(T_{f'}) = (T_f)'$

This defines the *distributional derivative.*

1. **Linearity:** $\langle T', a\varphi + b\psi \rangle = -\langle T, (a\varphi + b\psi)' \rangle = -\langle T, a\varphi' + b\psi' \rangle = -a\langle T, \varphi' \rangle - b\langle T, \psi' \rangle = a\langle T', \varphi \rangle + b\langle T', \psi \rangle$
2. **Continuity:** Suppose $\varphi_n \to \varphi$ in $\mathcal{D}$. Consider $\langle T', \varphi \rangle$.
$\langle T', \varphi_n \rangle = -\langle T, \varphi_n' \rangle \to -\langle T, \varphi' \rangle = \langle T', \varphi \rangle$, because T is continuous on $\mathcal{D}$ and $D : \varphi \to \varphi'$ is continuous on $\mathcal{D}$

Example 11.12.
Notes 1/31/11

$$f(x) = |x|, \quad |x| \le \pi$$
$$f'(x) = \operatorname{sgn} x = g(x)$$

Compute the distributional derivative of g:

$$\begin{aligned}
\langle g', \varphi \rangle &= -\langle g, \varphi' \rangle \\
&= -\int_0^\pi \varphi' \, dx + \int_{-\pi}^0 \varphi' \, dx \\
&= -[\varphi(\pi) - \varphi(0)] + [\varphi(0) - \varphi(\pi)] \\
&= 2\varphi(0) - 2\varphi(\pi) \\
&= 2\langle \delta_0, \varphi \rangle - 2\langle \delta_\pi, \varphi \rangle \\
&= \langle 2\delta_0 - 2\delta_\pi, \varphi \rangle
\end{aligned}$$

$$\begin{aligned}
g' &= 2\delta_0 - 2\delta_\pi \\
&= 2(\delta - \tau_\pi \delta)
\end{aligned}$$

Where τ_π means translation by π and δ_a is the δ-"function" supported at a:

$$\langle \delta_a, \varphi \rangle = \varphi(a)$$

Example 11.13.
Notes 1/31/11

Compute δ':

$$\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$$

Definition 11.14. ***Fourier Coefficients***
Notes 1/31/11

If $T \in \mathcal{D}'(\mathbb{T})$, define $\hat{T}(n) = \frac{1}{2\pi} \langle T, e^{-inx} \rangle$.

Example 11.15.
Notes 1/31/11

Compute the Fourier coefficients of δ:

$$\hat{\delta}(n) = \frac{1}{2\pi}\left\langle \delta, e^{-inx}\right\rangle = \frac{1}{2\pi}e^0 = \frac{1}{2\pi}$$

$$\delta(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{inx}$$

Remark 11.16.
1/31/11

There are 3 contexts in which to look at Fourier series:

- Continuous functions $\Rightarrow$ converge uniformly
- L^2 functions $\Rightarrow$ converge in L^2
- Distribution functions $\Rightarrow$ converge in the distributional sense

Example 11.17.
Notes 1/31/11

$$P_r(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|}e^{inx}$$

Formally, as $r \to 1^-$, $P_r(x) \rightharpoonup \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|}e^{inx} = \delta(x)$

Theorem 11.18.
Notes 1/31/11

$\varphi \in \mathcal{D}$ iff $(\hat{\varphi}(n))$ is rapidly decreasing, i.e.

$$|n|^k\hat{\varphi}(n) \to 0 \text{ as } n \to \infty \ \forall\ k \geq 0$$

and the Fourier series of φ converges to φ in $\mathcal{D}$.

Compare to Corollary 7.55.

Proof

- $\varphi \in C^k \Rightarrow |n|^k\hat{\varphi}(n) \to 0$ by the Riemann-Lebesgue Lemma, so if $\varphi \in C^\infty$, then the $\hat{\varphi}(n)$ are rapidly decreasing
- Sobolev Embedding Theorem: If $\hat{\varphi}(n)$ is rapidly decreasing, then $\varphi \in H^k(\mathbb{T})\ \forall\ k$ implies that

$\sum(1+n^2)|\hat{\varphi}(n)|^2 < \infty$

- Hence, $\varphi \in C^{k-1}(\mathbb{T})\ \forall\ k$. So $\varphi \in C^\infty$.
- Similarly, $\sum_{|n|\le N} \hat{\varphi}(n)e^{inx} \to \varphi$ in $H^k\ \forall\ k$
  - So $\sum_{|n|\le N} \hat{\varphi}(n)e^{inx} \to \varphi$ in $C^{k-1}\ \forall\ k$
  - So $\sum_{|n|\le N} \hat{\varphi}(n)e^{inx}$ converges in $\mathcal{D}$

Definition 11.19. $S(\mathbb{Z})$
Notes 2/2/11

$S(\mathbb{Z})$ is the space of rapidly decreasing sequences, (c_n), such that

$$\lim_{n\to\infty} |n|^k c_n = 0 \quad \forall\ k = 0, 1, 2, \ldots$$

Remark 11.20.
Notes 2/2/11

$\mathcal{F} : C^\infty(\mathbb{T}) \to S(\mathbb{Z})$
$\mathcal{F} : \varphi \to (\hat{\varphi}(n))$

If $\varphi \in C^\infty(\mathbb{T})$, then $S_N\varphi = \sum_{|n|\le N} \hat{\varphi}(n)e^{inx} \to \varphi$ in $\mathcal{D}$.

If $T \in \mathcal{D}'(\mathbb{T})$, then $\hat{T}(n) = \frac{1}{2\pi}\left\langle T, e^{-inx}\right\rangle$

Proposition 11.21.
Notes 2/2/11

$$\widehat{T'}(n) = in\hat{T}(n)$$

See Proposition 7.52 and Definition 7.56.

Proof.

$$\widehat{T'}(n) = \frac{1}{2\pi}\left\langle T', e^{-inx}\right\rangle = -\frac{1}{2\pi}\left\langle T, \left(e^{-inx}\right)'\right\rangle = in \cdot \frac{1}{2\pi}\left\langle T, e^{-inx}\right\rangle$$
$$= in\hat{T}(n)$$

$\square$

Definition 11.22. ***Slow Growth***
Notes 2/2/11

A sequence (c_n) has *slow growth* if there exist k, M such that $|c_n| \leq M(1+n^{2k})^{1/2}\ \forall\ n$.

Equivalently, $|c_n| \leq M|n|^k\ \forall\ n \neq 0$.

Lemma 11.23.
Notes 2/2/11

If $T \in \mathcal{D}'$, then $(\hat{T}(n))$ has slow growth.

Proof. If $T \in \mathcal{D}'$ then T has some finite order k such that

$$|\langle T, \varphi \rangle| \leq C\|\varphi\|_{C^k}$$

Then

$$|\hat{T}(n)| = |\langle T, e^{-inx}\rangle| \leq C\|e^{-inx}\|_{C^k} \leq C(1+n^{2k})^{1/2}$$

□

Example 11.24. ***Weierstrass Nowhwere Differentiable Function***
Notes 2/2/11

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\cos(3^n x)$$

$\sum \frac{1}{2^n} < \infty$, so $f \in \mathcal{A}(\mathbb{T})$.

$$f'(x) = \sum_{n=1}^{\infty} \frac{3^n}{2^n}\sin(3^n x)$$

f is nowhere differentiable, although it does have a distributional derivative.

Theorem 11.25.
Notes 2/2/11

If $T \in \mathcal{D}'(\mathbb{T})$ and $S_N T = \sum\limits_{|n|\leq N} \hat{T}(n)e^{inx} \in C^\infty(\mathbb{T})$, then $S_N T \rightharpoonup T$ in $\mathcal{D}'$ as $N \to \infty$.

Ex: $\delta(x) = \frac{1}{2\pi}\sum\limits_{n\in\mathbb{Z}} e^{inx}$

Proof.

$$\begin{aligned}\langle S_N T, \varphi\rangle &= \left\langle \sum_{|n|\le N} \hat{T}(n)e^{inx}, \varphi \right\rangle = \sum_{|n|\le N} \left\langle \hat{T}(n)e^{-inx}, \varphi\right\rangle = \sum_{|n|\le N} \hat{T}(n) \int e^{inx}\varphi(x)\,dx \\ &= 2\pi \sum_{|n|\le N} \hat{T}(n)\hat{\varphi}(-n) = \cancel{2\pi} \sum_{|n|\le N} \langle T, e^{-inx}\rangle \cdot \frac{1}{\cancel{2\pi}}\hat{\varphi}(-n) = \left\langle T, \sum_{|n|\le N} \hat{\varphi}(-n)e^{-inx} \right\rangle \\ &= \langle T, S_N\varphi\rangle \to \langle T, \varphi\rangle \text{ as } n \to \infty\end{aligned}$$

So $S_N T \to T$ as $N \to \infty$. ☐

Theorem 11.26.
Notes 2/2/11

If (c_n) is a sequence of slow growth, $(c_n) \in S'(\mathbb{Z})$, then there exists a distribution T such that $\hat{T}(n) = c_n$.

Proof. Define T by

$$\langle T, \varphi\rangle = 2\pi \sum_{n\in\mathbb{Z}} c_n \hat{\varphi}(-n)$$

☐

Remark 11.27.
Notes 2/2/11

$\mathcal{F} : f \mapsto \hat{f}(n)$

$$\mathcal{D}(\mathbb{T}) = C^\infty(\mathbb{T}) \leftrightarrow S(\mathbb{Z})$$

$$C(\mathbb{T}) \supset \mathcal{A}(\mathbb{T}) \leftrightarrow \ell'(\mathbb{Z})$$

$$L^2(\mathbb{T}) \leftrightarrow \ell^2(\mathbb{Z})$$

$$L^1(\mathbb{T}) \to C_0(\mathbb{Z})$$

$$\mathcal{D}'(\mathbb{T}) \leftrightarrow S'(\mathbb{Z})$$

- $C^\infty \subset L^2(\mathbb{T}) \subset \mathcal{D}'(\mathbb{T})$
- $S(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset S'(\mathbb{Z})$

8 Bounded Linear Operators on a Hilbert Space

8.1 Orthogonal Projections

Definition 8.1. ***Direct Sum***
page 187 and Notes 2/4/11

If M and N are subspaces of a linear space X such that every $x \in X$ can be written uniquely as $x = y + z$ with $y \in M$ and $z \in N$, then we say that $X = M \oplus N$ is the *direct sum* of M and N, and we call N a *complementary subspace* of M in X. The decomposition $x = y + z$ is unique if and only if $M \cap N = \{0\}$.

Definition 8.2. ***Projection, Idempotent, Self-Adjoint***
page 187 & 188 and Notes 2/4/11

Given a direct sum decomposition, $X = M \oplus N$, define the *projection* $P : X \to X$ onto M along N by

$$P(m + n) = m, \qquad m \in M, \quad n \in N$$

All projections are linear and *idempotent*, meaning that $P^2 = P$, because

$$P^2(m + n) = P(m) = m$$

Theorem 8.3.
page 188 and Notes 2/4/11

Any linear map $P : X \to X$ with $P^2 = P$ is a projection. Specifically, it is the projection onto $\operatorname{ran} P$ along $\ker P$.

Proof.

- $x = P(x) + (x - P(x))$
- $P^2(x) = P(x) \quad \Rightarrow \quad P(x) \in \operatorname{ran} P$
- $P(x - P(x)) = Px - P^2x = Px - Px = 0 \quad \Rightarrow \quad x - P(x) \in \ker P$
- Suppose $x \in \ker P \cap \operatorname{ran} P$
  - $x \in \operatorname{ran} P \quad \Rightarrow \quad x = Py$
  - $x \in \ker P \quad \Rightarrow \quad 0 = Px = P^2y = Py = x = 0$
  - Thus, $x = 0$, and $\ker P \cap \operatorname{ran} P = \{0\}$
- Thus, $X = \operatorname{ran} P \oplus \ker P$

□

Remark 8.4. ***Bounded Projections***
Notes 2/4/11

Question: Given a projection $P : X \to X$, X a Banach space, when can we say that P is bounded?

Answer: We need $\operatorname{ran} P$ closed and complemented by a closed subspace $N = \ker P$

Note: The kernel of a bounded operator is always closed; the range need not be.

Definition 8.5. ***Orthogonal Projections, Self-Adjoint***
Notes 2/4/11 and 2/7/11

Let $\mathcal{H}$ be a Hilbert space and let $M \subset \mathcal{H}$ be a closed linear subspace. Then by the Projection Theorem,

$$\mathcal{H} = M \oplus M^\perp, \qquad M^\perp = \{y \in \mathcal{H} \mid y \perp m \;\forall\, m \in M\}$$

We define the orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ onto M along $M^\perp$.

An *orthogonal projection* P on a Hilbert space $\mathcal{H}$ is

- **Idempotent:** $P^2 = P$
- ***Self-Adjoint:*** $\langle x, Py \rangle = \langle Px, y \rangle$

Proof. To see that a projection P on a Hilbert space $\mathcal{H}$ is self-adjoint, let

$$x = m + n, \qquad y = p + q, \qquad \text{where} \qquad m, p \in M, \qquad n, q \in N$$

Compute:

$$\langle x, Py \rangle = \langle m + n, p \rangle = \langle m, p \rangle + \langle n, p \rangle = \langle m, p \rangle$$
$$\langle Px, y \rangle = \langle m, p + q \rangle = \langle m, p \rangle + \langle m, q \rangle = \langle m, p \rangle$$

□

Lemma 8.6.
page 188 and Notes 2/7/11

If P is a nonzero othogonal projection then $\|P\| = 1$

Proof.

$$\|Px\|^2 = \langle Px, Px \rangle = \left\langle x, P^2 x \right\rangle = \langle x, Px \rangle \leq \|x\| \|Px\|$$

Either $\|Px\| = 0$ or $\|Px\| \leq \|x\|$. Since $\|Px\| \neq 0 \;\forall\, x$, it must be the case that $\|Px\| \leq \|x\|$. Then

$$\|P\| = \sup \frac{\|Px\|}{\|x\|} \leq 1$$

If $P \neq 0$, then there exists $y \in \mathcal{H}$ such that $Py \neq 0$. Setting $x = Py$ in the previous equation yields

$$\|P\| \geq \frac{\|P \cdot Px\|}{\|Px\|} = 1$$

So $\|P\| = 1$. □

Theorem 8.7.
page 189 and Notes 2/7/11

If P is an orthogonal projection, then $\mathcal{H} = M \oplus M^{\perp} = \operatorname{ran} P \oplus \ker P$, where $M = \operatorname{ran} P$ and $M^{\perp} = \ker P$ are closed subspaces. Conversely, if M is any closed subspace of $\mathcal{H}$, then there exists an orthogonal projection with $M = \operatorname{ran} P$ and $M^{\perp} = \ker P$.

Example 8.8. ***Even & Odd Functions***
page 189 and Notes 2/7/11

Let $\mathcal{H} = L^2(\mathbb{R})$ and let

$$M = \text{space of even functions, } f(-x) = f(x)$$
$$N = \text{space of odd functions, } f(-x) = -f(x)$$

$M \perp N$, since $\int \overline{f} g \, dx = 0$ for f odd, g even. Define

- **Even Projection:** $P : \mathcal{H} \to \mathcal{H}$ onto M, $Pf(x) = \frac{1}{2}[f(x) + f(-x)]$
- **Odd Projection:** $Q : \mathcal{H} \to \mathcal{H}$ onto N, $Qf(x) = \frac{1}{2}[f(x) - f(-x)]$
  - Note: $Q = I - P$

Check that P is self-adjoint:

$$\langle Pf, g\rangle = \int_{\mathbb{R}} \frac{1}{2}\overline{[f(x) + f(-x)]} g(x) \, dx = \int_{\mathbb{R}} \frac{1}{2}\overline{f}(x)g(x) + \frac{1}{2}\overline{f}(x)g(-x) \, dx = \langle f, Pg\rangle$$

Example 8.9.
Notes 2/7/11

Let $\mathcal{H} = L^2(\mathbb{T})$. Define $Pf = \frac{1}{2\pi}\int_{\mathbb{T}} f \, dx$, $P : \mathcal{H} \to \mathcal{H}$.

$$\text{Given: } f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}$$
$$\text{Then: } Pf = \hat{f}(0)$$

- **Idempotent:** $P^2 = P$ since Pf is a constant, and $P1 = 1$
- **Self-Adjoint:** $\langle Pf, g\rangle = \int \overline{\left[\frac{1}{2\pi}\int f \, dx\right]} g \, dx = \frac{1}{2\pi}\int \overline{f} \, dx \int g \, dx = \langle f, 1\rangle \cdot \frac{1}{2\pi}\int g \, dx = \langle f, Pg\rangle$

$\operatorname{ran} P$ = constant functions $=< 1 >$ (space spanned by 1)
$\ker P$ = functions with zero mean (i.e. $\hat{f}(0) = 0$)
$\operatorname{ran} P \perp \ker P$

Example 8.10. ***Fourier Projections***
Notes 2/7/11

We can define the orthogonal projection of f onto the Nth partial sum of its Fourier series:

$$P_N f = \sum_{|n| \leq N} \hat{f}(n) e^{inx}$$

Similarly, we can define the projection onto the positive n part of its Fourier series:

$$Pf = \sum_{n=0}^{\infty} \hat{f}(n) e^{inx}$$

$$(I - P)f = \sum_{n=-\infty}^{-1} \hat{f}(n) e^{inx}$$

Example 8.11.
page 189 and Notes 2/7/11

Let $\mathcal{H} = L^2(\mathbb{R})$. If $A \subset \mathbb{R}$ is some Lebesgue measurable set, define

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then

$$P_A f = \chi_A f$$

is an orthogonal projection of $L^2(\mathbb{R})$ onto the subspace of functions with support contained in $\overline{A}$.

8.2 The Dual of a Hilbert Space

Theorem 8.12. ***Riesz Representation Theorem***
page 191 and Notes 2/7/11

Given: a Hilbert space $\mathcal{H}$, its dual space $\mathcal{H}^* = \mathcal{B}(\mathcal{H}, \mathbb{C})$ (the set of bounded linear maps $\varphi : \mathcal{H} \to \mathbb{C}$ with $\|\varphi\|_{\mathcal{H}^*} = \sup \frac{|\varphi(x)|}{\|x\|} < \infty$).

Every $\varphi \in \mathcal{H}^*$ can be given by $\varphi(x) = \langle y, x \rangle$ for some $y \in \mathcal{H}$, and $\|\varphi\| = \|y\|$. Conversely, every $y \in \mathcal{H}$ corresponds to a $\varphi \in \mathcal{H}^*$. The map $J : \varphi \mapsto y$ is an isometric, antilinear isomorphism of $\mathcal{H}^*$ onto $\mathcal{H}$.

$$\text{Antilinear:} \quad J(\varphi + \psi) = J(\varphi) + J(\psi)$$

$$J(\lambda\varphi) = \overline{\lambda} J(\varphi)$$

Proof.

- Suppose $\varphi \in \mathcal{H}^*$. We want to find $y \in \mathcal{H}$ such that $\varphi(x) = \langle y, x \rangle$

- Suppose $\varphi \neq 0$. Then $\ker\varphi \neq \mathcal{H}$ and $\ker\varphi$ is closed because φ is bounded
- There exists $z \in (\ker\varphi)^\perp$ (by the Projection Theorem)
- Consider $P : \mathcal{H} \to \mathcal{H}$, $Px = \frac{\varphi(x)}{\varphi(z)} Pz$. Claim: this is an orthogonal projection.
  - **Idempotent:** $P^2 x = P\left(\frac{\varphi(x)}{\varphi(z)} z\right) = \frac{\varphi(x)}{\varphi(z)} Pz = \frac{\varphi(x)}{\varphi(z)} z \quad$ (since $Pz = z$)
  - **Self-Adjoint:** $\langle x, Py\rangle = \langle Px, y\rangle$
- $\mathcal{H} = \operatorname{ran} P \oplus \ker P, \quad \operatorname{ran} P =< z >, \quad \ker P = \ker\varphi$
- $x \in \mathcal{H}, \quad x = \alpha z + w, \quad w \in \ker\varphi, \quad \alpha = \frac{\langle z,x\rangle}{\|z\|^2}$
- $\varphi(x) = \alpha\varphi(z) = \frac{\langle z,x\rangle}{\|z\|^2}\varphi(z) = \langle y, x\rangle, \quad y = \frac{\overline{\varphi}(z)}{\|z\|^2} z$

□

8.3 The Adjoint of an Operator

Definition 8.13. ***Adjoint***
page 193 and Notes 2/9/11

Given a bounded linear map $A \in \mathcal{B}(\mathcal{H})$, its *adjoint* $A^* \in \mathcal{B}(\mathcal{H})$ (← proved in Proposition 8.15) is the linear map that satisfies

$$\langle x, Ay\rangle = \langle A^*x, y\rangle \quad \forall\, x, y \in \mathcal{H}$$

Remark 8.14. ***Adjoint: Existence and Uniqueness***
page 193 and Notes 2/9/11

To define A^* such that$A^*x = z$, consider $\varphi_x : \mathcal{H} \to \mathbb{C}$, $\varphi_x(y) = \langle x, Ay\rangle$. Then

$$\|\varphi_x(y)\| \leq \|x\|\|Ay\| \leq \|x\|\|A\|\|y\|$$
$$\|\varphi_x\| \leq \|A\|\|x\|$$

So φ_x is a bounded linear functional. By the Riesz Representation Theorem, there is a unique $z \in \mathcal{H}$ such that

$$\varphi_x(y) = \langle z, y\rangle$$

Define $A^*x = z$. Then

$$\langle x, Ay\rangle = \varphi_x(y) = \langle z, y\rangle = \langle A^*x, y\rangle$$
$$\langle x, Ay\rangle = \langle A^*x, y\rangle \quad \forall\, x, y \in \mathcal{H}$$

Proposition 8.15.
Notes 2/9/11

If $A \in \mathcal{B}(\mathcal{H})$ then $A^* \in \mathcal{B}(\mathcal{H})$ and

$$
\begin{aligned}
&(1) \quad \|A^*\| = \|A\| \\
&(2) \quad \|A\|^2 = \|A^*A\|
\end{aligned}
$$

(See also Corollary 8.34.)

Proof.

$$
\begin{aligned}
\|A^*\| &= \sup_{\|x\|=1} \|A^*x\| \qquad \text{(See Lemma 8.26 in the book)} \\
&= \sup_{\|x\|=\|y\|=1} |\langle y, A^*x\rangle| = \sup_{\|x\|=\|y\|=1} |\langle Ay, x\rangle| = \sup_{\|y\|=1} \|Ay\| = \|A\|
\end{aligned}
$$

$$
\begin{aligned}
\|A\|^2 &= \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} |\langle Ax, Ax\rangle| = \sup_{\|x\|=1} |\langle x, A^*Ax\rangle| \\
&\leq \|A^*A\| \qquad \text{(See Corollary 8.27 in the book)}
\end{aligned}
$$

$$\|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2$$

$$\|A^*A\| = \|A\|^2$$

□

Remark 8.16.
Notes 2/9/11

$\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

$$\|AB\| \leq \|A\|\|B\| \qquad * : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),\ ** = \text{identity} \qquad \|A^*\| = \|A\|$$

Remark 8.17. ***Generalizations***
Notes 2/9/11

1. Given: $A : \mathcal{H} \to K,\ A^* : K \to \mathcal{H}$, where $\mathcal{H}, K$ are Hilbert spaces.
   $\langle x, Ay\rangle_K = \langle A^*x, y\rangle_H\ \ \forall\ y \in \mathcal{H},\ x \in K$
   A^* is the Hilbert space adjoint.
2. Given: $A : X \to Y,\ A' : Y' \to X'$, where X, Y are Banach spaces and X' is the dual space of X.
   $\langle \psi, Ax\rangle_{Y \times Y'} = \langle A'\psi, x\rangle_{X \times X'}\ \ \forall x \in X,\ \psi \in Y'$
   A' is the dual operator or Banach space adjoint.

Example 8.18.
page 193 and Notes 2/9/11 and Notes 2/11/11

Let $\mathcal{H} = \mathbb{C}^n$. Then $A : \mathbb{C}^n \to \mathbb{C}^n$ is given by a matrix (a_{ij}).

$$\begin{aligned}
y_i &= \sum_{j=1}^{n} a_{ij} x_j, \qquad x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n) \\
\langle x, y \rangle &= \sum_{i=1}^{n} x_i y_i \\
\langle x, Ay \rangle &= \sum_{i=1}^{n} \overline{x_i} \left(\sum_{j=1}^{n} a_{ij} y_j \right) = \sum_{j=1}^{n} \left(\overline{\sum_{i=1}^{n} \overline{a_{ij}} x_i} \right) y_j \\
&= \langle A^* x, y \rangle
\end{aligned}$$

$$\begin{aligned}
&\text{If } z = A^* x \\
&z_j = \sum_{i=1}^{n} \overline{a_{ij}} x_i = \sum_{j=1}^{n} \overline{a_{ji}} x_j
\end{aligned}$$

- A^* has matrix $(\overline{a_{ji}})$, which is the conjugate transpose of (a_{ij})
- (A^*A) is Hermitian, positive definite
- $(A^*A)^* = (A^*A)^* = A^*A$
- $\langle x, A^*Ax \rangle = \langle Ax, Ax \rangle \geq 0$
- A^*A has orthogonal eigenvectors that form a basis of $\mathbb{C}^n$ with eigenvalues $\mu_1, \mu_2, \dots, \mu_n \geq 0$
- $\|A^*A\| = \max_{1 \leq j \leq n} |\mu_j| = \sigma(A^*A) =$ the spectral radius of A^*A
- $\|A\| = \sqrt{\sigma(A^*A)}$

Example 8.19.
page 194 and Notes 2/9/11

Let $\mathcal{H} = L^2([0,1])$, $\langle f, g\rangle = \int_0^1 \overline{f(x)} g(x)\,dx$.
Define the integral operator $K : L^2([0,1]) \to L^2([0,1])$ by

$$Kf(x) = \int_0^1 k(x,y) f(y)\,dy, \qquad k : [0,1]\times[0,1] \to \mathbb{C}$$

(Note: $k(x,y)$ is the kernel of the integral operator K. It is not related to the null space.)
Ex: Assume that k is *Hilbert-Schmidt*: k is measurable on $[0,1]\times[0,1]$ and

$$\|K\|^2 \le \int_0^1 \int_0^1 |k(x,y)|^2\,dx\,dy < \infty$$

$$\begin{aligned}\langle f, Kg\rangle &= \int_0^1 \overline{f(x)} \left(\int_0^1 k(x,y) g(y)\,dy\right) dx \\ &= \int_0^1 \left(\overline{\int_0^1 f(x)\overline{k(x,y)}\,dx}\right) g(y)\,dy \\ &= \langle K^* f, g\rangle\end{aligned}$$

Since

$$\begin{aligned}K^* f(y) &= \int_0^1 \overline{k(x,y)} f(y)\,dx \\ K^* f(x) &= \int_0^1 \overline{k(y,x)} f(y)\,dy\end{aligned}$$

Thus, K^* is an integral operator with conjugate transpose level of k.

Example 8.20.
page 194 and Notes 2/9/11

Recall the right and left shift operators, respectively:

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots) \qquad T(x_1, x_2, x_3 \dots) = (x_2, x_3, \dots)$$

T is the adjoint of S, i.e. $T = S^*$. Also, $S = T^*$.

Example 8.21. ***Solvability of Linear Equations***
Notes 2/11/11

Consider $A : \mathcal{H} \to \mathcal{H}$, $Ax = y$. Suppose for some $y \in \mathcal{H}$ we have a solution for $x \in \mathcal{H}$.

Let $z \in \ker A^*$. Then

$$\langle z, Ax\rangle = \langle A^* z, x\rangle = \langle z, y\rangle$$

Thus, a necessary condition for solvability is that $y \perp z \;\forall\; z \in \ker A^*$, i.e. $y \perp \ker A^*$.

Theorem 8.22.
page 194 and Notes 2/11/11

If $A \in \mathcal{B}(\mathcal{H})$, then $\mathcal{H} = \overline{\operatorname{ran} A} \oplus (\ker A^*)$, and

$$\overline{\operatorname{ran} A} = (\ker A^*)^\perp \qquad \ker A = (\operatorname{ran} A^*)^\perp$$

Proof. From Example 8.21, if $y \in \operatorname{ran} A$ then $y \in (\ker A^*)^\perp$.

$$\operatorname{ran} A \subset (\ker A^*)^\perp$$
$$\overline{\operatorname{ran} A} \subset (\ker A^*)^\perp \qquad \text{since orthogonal complements are closed}$$

If $y \in (\operatorname{ran} A)^\perp$ then

$$\langle Ax, z\rangle = 0 \;\forall\; x \in \mathcal{H}$$
$$\langle x, A^*y\rangle = 0 \;\forall\; x \in \mathcal{H}$$

This implies that $A^*y = 0$, so $y \in \ker A^*$.

$$(\operatorname{ran} A)^\perp \subset \ker A^*$$
$$\overline{\operatorname{ran} A} = (\operatorname{ran} A)^{\perp\perp} \supset (\ker A^*)^\perp$$

□

Corollary 8.23.
page 195 and Notes 2/11/11

If $A \in \mathcal{B}(\mathcal{H})$ has closed range ($\operatorname{ran} A$ is a closed linear subspace), then $Ax = y$ is solvable iff $y \perp \ker A^*$.

Example 8.24.
Notes 2/11/11

If $\mathcal{H}$ is finite dimensional, or A has finite rank, then $\operatorname{ran} A$ is closed and Corollary 8.23 applies.

Example 8.25.
page 196 and Notes 2/11/11

Recall the left (T) and right (S) shift operators. $S^* = T$, $T^* = S$.

1. $\mathcal{H} = \overline{\operatorname{ran} S} \oplus \ker S^* = \overline{\operatorname{ran} S} \oplus \ker T$
2. $\mathcal{H} = \overline{\operatorname{ran} T} \oplus \ker T^* = \overline{\operatorname{ran} T} \oplus \ker S$

- $\operatorname{ran} S = \left\{(x_1, x_2, \ldots) \in \ell^2 \;\middle|\; x_1 = 0\right\}$
- $\operatorname{ran} T = \ell^2(\mathbb{N})$
- $\ker S = \{0\}$
- $\ker T = \left\{(x_1, 0, 0, 0, \ldots) \;\middle|\; x_1 \in \mathbb{C}\right\}$

$Sx = y$ is solvable iff $y \perp \ker T$, and the solution is unique.
$Tx = y$ is solvable for all $y \in \ell^2(\mathbb{N})$, but the solution is not unique.

Example 8.26.
Notes 2/11/11

Let $\mathcal{H} = \ell^2(\mathbb{N})$, $A(x_1, x_2, \dots, x_n, \dots) = (x, \frac{1}{2}x_2, \dots, \frac{1}{n}x_n, \dots)$.

$$[A] = \begin{pmatrix} 1 & & & & & \\ & \frac{1}{2} & & & & \\ & & \frac{1}{3} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{n} & \\ & & & & & \ddots \end{pmatrix}, \qquad A^* = A \text{ (self-adjoint)}$$

$\ker A = \ker A^* = \{0\}$
$\mathcal{H} = \overline{\operatorname{ran} A} \oplus \ker A$

Given $y = (y_1, y_2, \dots) \in \ell^2(\mathbb{N})$, does there exist $x = (x_1, x_2, \dots) \in \ell^2(\mathbb{N})$ such that $Ax = y$?

$x \in \ell^2(\mathbb{N}) \Leftrightarrow \sum n^2 |y_n| < \infty$
$\operatorname{ran} A = \{(x_1, x_2, \dots) \in \ell^2(\mathbb{N}) \mid \sum n^2 |x_n|^2 < \infty\}$

$\operatorname{ran} A \neq \mathcal{H}$, so A is not onto.
Ex: $M = \{(x_1, x_2, \dots, x_N, 0, 0, \dots)\} \subset \operatorname{ran} A$
M is dense in $\ell^2(\mathbb{N})$, so $\overline{\operatorname{ran} A} = \ell^2(\mathbb{N})$, $\quad \ell^2(\mathbb{N}) = \overline{\operatorname{ran} A} \oplus \ker A^*$

Consider: $Ax = y$, $y \perp \ker A^* = \ker A = \{0\}$ $\forall\, y \in \ell^2(\mathbb{N})$. This is not solvable for every $y \in \ell^2(\mathbb{N})$, only for $y \in \operatorname{ran} A$, and $\operatorname{ran} A$ is a dense, non-closed subspace of $\ell^2(\mathbb{N})$.

8.4 Self-Adjoint and Unitary Operators

Definition 8.27. ***Self-Adjoint***
page 197 and 2/14/11

A bounded operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is *self-adjoint* if $A^* = A$.

Equivalently, A is *self-adjoint* iff

$$\langle x, Ay \rangle = \langle Ax, y \rangle \qquad \forall\, x, y \in \mathcal{H}$$

Example 8.28. ***Self-Adjoint Operators***
Notes 2/14/11

1. $A : \mathbb{C}^n \to \mathbb{C}^n$, $[A]^* = [A]$
   $A : \mathbb{R}^n \to \mathbb{R}^n$, $[A]^T = [A]$
2. $\mathcal{H} = L^2(\mathbb{R})$. Suppose $a : \mathbb{R} \to \mathbb{C}$ is bounded and measurable. Define $M : \mathcal{H} \to \mathcal{H}$, $Mf = af$.
   $\|Mf\|_2 \leq \|a\|_\infty \|f\|_2$.
   $M^* f = \overline{a} f$, $M^* = M$ if $a : \mathbb{R} \to \mathbb{R}$.
3. Orthogonal projections: $P^2 = P = P^*$ (self-adjoint)
4. Given $T \in \mathcal{B}(\mathcal{H})$, $A = T^*T$ is self-adjoint.
   $T = A + iB$, $A = \frac{1}{2}(T^* + T)$, $B = \frac{1}{2i}(T^* - T)$
   $A^* = A$, $B^* = B$
5. The shift operators are NOT self-adjoint because $S^* = T \neq S$

Definition 8.29. ***Bilinear Forms, Sesquilinear***
page 197 and Notes 2/14/11

Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. We define the *bilinear form* $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$a(x, y) = \langle x, Ay \rangle$$

We say that a is *sesquilinear* because

$$a(x, \lambda y + \mu z) = \lambda a(x, y) + \mu a(x, z)$$
$$a(\lambda x + \mu y, z) = \overline{\lambda} a(x, z) + \overline{\mu} a(x, z)$$

Definition 8.30. ***Hermitian Symmetric & Symmetric***
page 197 and Notes 2/14/11

Suppose A is self-adjoint. Then

$$\langle x, Ay \rangle = \langle Ax, y \rangle = \overline{\langle y, Ax \rangle}$$
$$a(x, y) = \overline{a(x, y)}$$

We say that a is *Hermitian symmetric*. In the real case, we have $a(x, y) = a(y, x)$, and we say that this is *symmetric*.

Definition 8.31. ***Quadratic Form***
page 197 and Notes 2/14/11

Given $A : \mathcal{H} \to \mathcal{H}$, we define the *quadratic form* $q : \mathcal{H} \to \mathbb{C}$ by

$$q(x) = \langle x, Ax \rangle = a(x, x)$$

If A is self-adjoint, then $a(x, x) = \overline{a(x, x)}$, so $a(x, x)$ is real for all $x \in \mathcal{H}$.

Definition 8.32. ***Positive, Positive Definite***
page 198 and Notes 2/14/11

A self-adjoint operator A is *positive* or *positive definite* if $\langle x, Ax\rangle = a(x,x) > 0$ for all $x \in \mathcal{H}, x \neq 0$.

Theorem 8.33.
page 198 and Notes 2/14/11

If A is self-adjoint then

$$\|A\| = \sup_{x\neq 0} \frac{|\langle x, Ax\rangle|}{\|x\|^2} = \sup_{\|x\|=1} |\langle x, Ax\rangle|$$

Note: compare this to $\|A\| = \sup\limits_{\|x\|=1} |\langle Ax, Ax\rangle|^{1/2}$ (see part 2 of Proposition 8.15).

Proof.

$$|\langle x, Ax\rangle| \leq \|x\|\|Ax\| \leq \|A\|\|x\|^2 \qquad \text{(Cauchy-Schwarz)}$$

Let $\alpha = \sup\limits_{\|x\|\neq 0} \frac{|\langle x,Ax\rangle|}{\|x\|^2} \leq \|A\|$. Then $|\langle x, Ax\rangle| \leq \alpha\|x\|^2 \leq \|A\|\|x\|^2$. The parallelogram law states that

$$\langle x, Ay\rangle = \frac{1}{4}\{ \langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle - i\langle x+iy, A(x+iy)\rangle + i\langle x-iy, A(x-iy)\rangle \}$$

In general,

$$\|A\| = \sup_{\|x\|=\|y\|=1} |\langle x, Ay\rangle|$$

and this does not require self-adjoint. If A is self-adjoint, the first 2 terms in the parallelogram law expression are real and the last 2 are imaginary. We can multiply y by $e^{i\theta}$ so that $e^{i\theta}\langle x, Ay\rangle = \langle x, Az\rangle$ is real, where $z = ye^{i\theta}$. Then we have

$$\begin{aligned}
e^{i\theta}\langle x, Ay\rangle &= \langle x, Az\rangle \\
&= \frac{1}{4}\{ \langle x+z, A(x+z)\rangle - \langle x-z, A(x-z)\rangle \} \\
|\langle x, Ay\rangle| &\leq \frac{1}{4}|\langle x+z, A(x+z)\rangle| + \frac{1}{4}|\langle x-z, A(x-z)\rangle| \\
&\leq \frac{\alpha}{4}\left(\|x+z\|^2 + \|x-z\|^2\right) \\
&\leq \frac{\alpha}{2}\left(\|x\|^2 + \|z\|\right) \qquad \text{(by the parallelogram rule (not law))} \\
\|A\| \leq \sup_{\|x\|=\|y\|=1} |\langle x, Ay\rangle| &\leq \frac{\alpha}{2}(\|x\|^2 + \|y\|^2) \leq \frac{\alpha}{2}(1+1) = \alpha
\end{aligned}$$

□

Corollary 8.34.
page 199

If A is a bounded operator on a Hilbert space then $\|A^*A\| = \|A\|^2$. If A is self-adjoint, then $\|A^2\| = \|A\|^2$.

The proof follows directly from Proposition 8.15.

Definition 8.35. ***Unitary Operators***
pages 199 & 200 and Notes 2/14/11

An operator $U : \mathcal{H} \to \mathcal{H}$ is *unitary* if

$$U^*U = UU^* = I, \qquad \text{i.e. } U^* = U^{-1}$$

Note that

$$\langle Ux, Uy \rangle = \langle U^*Ux, y \rangle = \langle x, y \rangle$$

so U preserves norms and inner products. Furthermore, if $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$, then so is $\{Ue_n \mid n \in \mathbb{N}\}$.

Example 8.36.
Notes 2/14/11

1. $U : \mathbb{C}^2 \to \mathbb{C}^2$ with matrix

$$[U] = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b^2| = 1, \quad a, b \in \mathbb{C}$$

   In the real case, $a = \cos\theta$, $b = \sin\theta$, and U is rotation by θ.
2. The right shift operator S on $\ell^2(\mathbb{N})$ is not unitary because

$$S^* = T, \qquad S^*S = I, \qquad SS^* = P \neq I$$

3. If $A^* = A$ then $U = e^{iA}$ is unitary, where

$$\begin{aligned} e^{iA} &= I + (iA) + \cdots + \frac{1}{n!}(iA)^n + \dots \\ U^* &= e^{-iA} \\ U^*U &= I \end{aligned}$$

Example 8.37. ***Quantum Mechanics***
Notes 2/14/11

In quantum mechanics we have the Hamiltonian operator H, with $H^* = H$. We also have $U(t) = e^{itH}$, $U : \mathcal{H} \to K$, $U^* : K \to \mathcal{H}$. U is unitary if $U^*U = I_H$ and $UU^* = I_K$. We say that 2 Hilbert spaces are *isometric* if they are unitarily equivalent.

Example 8.38.
page 201 and Notes 2/14/11

$$\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}) \quad \text{is unitary}$$
$$\mathcal{F}f = \hat{f}, \quad \hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx}\, dx$$

Definition 8.39. ***Normal Operators***
Notes 2/16/11

If $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$, then T is *normal* if

$$[T^*, T] \equiv T^*T - TT^* = 0 \qquad \text{i.e.} \qquad T^*T = TT^*$$

Self-adjoint and unitary operators are normal.

Example 8.40.
Notes 2/16/11

1. Self-adjoint and unitary operators are normal
2. The shift operators on $\ell^2(\mathbb{N})$ are not normal
3. Any multiplication operator is normal

$$\begin{aligned} &M : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\ &(Mf)(x) = m(x)f(x), \qquad m \in L^\infty(\mathbb{R}) \\ &M^*f = \overline{m}f \\ &M^*Mf = \overline{m}mf = m\overline{m}f = MM^*f \end{aligned}$$

Special cases
(a) If m is real-valued then $M = M^*$, so M is self-adjoint. For
(b) For M to be unitary, we must have $m = e^{i\theta}$.

8.6 Weak Convergence in a Hilbert Space

Definition 8.41. ***Weak Convergence***
page 204 and Notes 2/16/11

A sequence (x_n) in a Hilbert space $\mathcal{H}$ *converges weakly* to $x \in \mathcal{H}$, written $x_n \rightharpoonup x$, if

$$\langle x_n, y \rangle \to \langle x, y \rangle \quad \forall\, y \in \mathcal{H}$$

Compare to Distributional Convergence (Definition 11.2): $T_n \rightharpoonup T$ in $\mathcal{D}'$ if $\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle$.

Definition 8.42. ***Strong Convergence***
Notes 2/16/11

We write *strong (norm) convergence* as $x_n \to x$ if $\|x_n - x\| \to 0$.

Remark 8.43. ***Weak vs. Strong Convergence***
Notes 2/16/11

If $x_n \to x$, then $x_n \rightharpoonup x$ because

$$|\langle x_n, y\rangle - \langle x, y\rangle| \le \|x_n - x\|\|y\| \qquad \text{(Cauchy-Schwarz)}$$

In a finite dimensional space, the converse is true, but this is not the case in infinite dimensional spaces.

Weak convergence = component-wise convergence

Example 8.44.
page 204 and Notes 2/16/11

Let $\mathcal{H}$ be a separable Hilbert space and let $\{e_n \mid n \in \mathbb{N}\}$ be a separable orthonormal basis. Then $e_n \rightharpoonup 0$ as $n \to \infty$ because

$$\langle e_n, y\rangle = y_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{because} \quad \sum |y_n|^2 < \infty$$

But (e_n) doesn't converge strongly because

$$\|e_n - e_m\| = \sqrt{2} \quad \forall\, n \neq m$$

and so the sequence is not Cauchy and hence not convergent.

Example 8.45.
Notes 2/16/11

Define an unbounded sequence (x_n) by $x_n = ne_n$. We know that

$$\langle x_n, e_m\rangle \to 0 \quad \Rightarrow \quad \langle x_n, y\rangle \to 0 \quad \text{as} \quad n \to \infty \quad \forall\, y = \sum_{m=1}^{m} c_m e_m$$

Let $y_1 = \sum \frac{1}{m} e_m$. Then

$$\langle x_n, y\rangle = \frac{1}{n} \cdot n = 1 \quad \forall\, n$$

Let $y_2 = \sum \frac{1}{m^{3/4}} e_m \in \mathcal{H}$. Then

$$\langle x_n, y\rangle = \frac{1}{n^{3/4}} \cdot n \to 0$$

Thus, (x_n) does not converge weakly.

Theorem 8.46. ***Uniform Boundedness Theorem***
page 204

Suppose that $\{\varphi_n : X \to \mathbb{C} \bigm| n \in \mathbb{N}\}$ is a set of functionals on a Banach space X such that the set of complex numbers $\{\varphi_n(x) \bigm| n \in \mathbb{N}\}$ is bounded for each $x \in X$. Then $\{\|\varphi_n\| \bigm| n \in \mathbb{N}\}$ is bounded.

Theorem 8.47.
Notes 2/16/11

If $x_n \rightharpoonup x$ then $\{\|x_n\| \bigm| n \in \mathbb{N}\}$ is bounded.

Proof. Define $\varphi_n : \mathcal{H} \to \mathbb{C}$ by $\varphi_n(y) = \langle x_n, y\rangle$. Then $\varphi_n \in \mathcal{H}^*$. By the uniform boundedness theorem (Theorem 8.46),

$$|\varphi_n(y)| \le M \quad \forall\, y \in \mathcal{H}, n \in \mathbb{N}$$
$$\{|\varphi_n(y)| \bigm| n \in \mathbb{N}\} \text{ is bounded for each } y \in \mathcal{H}, \text{ so } \{\|\varphi_n\| \bigm| n \in \mathbb{N}\} \text{ is bounded}$$

□

Theorem 8.48.
page 205 and Notes 2/16/11

Let $D \subset \mathcal{H}$ be a dense subset. Then $x_n \rightharpoonup x$ iff

(a) $\{\|x_n\| \bigm| n \in \mathbb{N}\}$ is bounded

(b) $\langle x_n, y\rangle \to \langle x, y\rangle \quad \forall\, y \in D$

Proposition 8.49.
page 208 and Notes 2/16/11

If $x_n \rightharpoonup x$, then $\|x\| \le \liminf_{n\to\infty} \|x_n\|$

Proof.

$$\|x\|^2 = \langle x, x\rangle = \lim_{n\to\infty} \langle x_n, x\rangle \le \|x\| \liminf_{n\to\infty} \|x_n\|$$
$$\langle x_n, x\rangle \le \|x_n\|\|x\| \qquad \text{(Cauchy-Schwarz)}$$

Note: if $a_n \le b_n$, $a_n \to a$, then $a \le \liminf b_n$.

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \|x_n\|^2 - \langle x, x_n\rangle - \langle x_n, x\rangle + \|x\|^2$$

If $x_n \rightharpoonup x$, then $\|x_n\| \to \|x\|$, and

$$\|x_n - x\|^2 \to \|x\|^2 - \langle x, x\rangle - \langle x, x\rangle + \|x\|^2 = 0$$

□

Example 8.50. ***Example for Proposition 8.49***
Notes 2/16/11

$$x_1 = e_1 \qquad x_n \rightharpoonup 0$$
$$x_2 = 2e_2 \qquad \|x_n\| = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$
$$x_3 = e_3$$
$$x_4 = 2e_4 \qquad \liminf_{n\to\infty} = 1$$
$$\dots$$

Example 8.51. ***Weak Convergence $\not\Rightarrow$ Strong Convergence***
Notes 2/16/11

(a) **Oscillation:**

(1) Let $\mathcal{H} = L^2(\mathbb{T}),\ f_n(x) = e^{inx} \rightharpoonup 0$ as $n \to \infty$

Proof. $\|f_n\| = \sqrt{2\pi}$ is bounded, and $\langle e^{inx}, \varphi \rangle \to 0$ as $n \to \infty$ for all trig polynomials φ, and the trig polynomials are dense in $L^2(\mathbb{T})$. $\square$

(2) Let $\mathcal{H} = L^2(\mathbb{R})$. Recall that $C_C^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ are the smooth functions with compact support, and they are dense in $L^2(\mathbb{R})$. Then $f_n \rightharpoonup f$ iff

i. $\|f\| \le M$ (bounded)

ii. $\int f_n \varphi\, dx \to \int f \varphi\, dx\ \forall\ \varphi \in C_C^\infty(\mathbb{R})$

Consider $f_n(x) = \psi(x)\sin(n\pi x)$, where $\psi \in C_C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $f_n \rightharpoonup 0$ as $n \to \infty$, but $f_n \not\to 0$ as $n \to \infty$. (See proof below)

(b) **Concentration:** Consider

$$f_n(x) = \begin{cases} n^{1/2} & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

i. $\|f_n\|^2 = \int_0^{1/2} \left(n^{1/2}\right)^2\, dx = 1$

ii. $\forall\ \varphi \in C_C^\infty(\mathbb{R}), \quad \left|\int f_n \varphi\, dx\right| = \left|n^{1/2} \int_0^{1/n} \varphi\, dx\right| \le n^{1/2} \cdot \frac{1}{n} \|\varphi\|_\infty \to 0$ as $n \to \infty$

So $f_n \rightharpoonup 0$ as $n \to \infty$

Does f_n converge strongly to 0? No, because $\|f_n\| = 1\ \forall\ n$. (See below for more details)

(c) **Escape to Infinity:**

$$f_n(x) = \begin{cases} 1 & n < x < n+1 \\ 0 & \text{otherwise} \end{cases}$$

i. $\|f_n\|_{L^2} = 1$, so f_n is bounded.

ii. $\int f_n \varphi\, dx \to 0$ as $n \to \infty\ \forall \varphi \in C_C^\infty(\mathbb{R})$

Thus, $f_n \rightharpoonup 0$, but $f_n \not\to 0$ because $\|f_n\| = 1\ \forall\ n$.

Proof. (a2)

i. $\|f_n\|^2 = \int \psi^2(x)\sin^2(n\pi x)\, dx \le \int \psi^2(x)\, dx \le \|\psi\|^2$

ii. Suppose $\varphi \in C_C(\mathbb{R})$.

$$\int f_n(x)\varphi(x)\,dx = \int \psi(x)\sin(n\pi x)\varphi(x)\,dx$$
$$= \int \frac{\cos(n\pi x)}{n\pi}\left[\varphi(x)\psi(x)\right]'\,dx \qquad \text{(IBP, no boundary terms because } \varphi \in C_C(\mathbb{R})\text{)}$$
$$\left|\int f_n\varphi\,dx\right| \le \frac{1}{n\pi}\int (|\varphi\psi|)'\,dx$$
$$\le \frac{c}{n}$$

So $\int f_n\varphi\,dx \to 0$ as $n \to \infty$, and thus $f_n \rightharpoonup f$.

Does (f_n) converge strongly? i.e., does $f_n \to 0$? (see Remark 8.52)
If $\psi \neq 0$, then

$$\|f_n\|^2 = \int \psi^2(x)\sin^2(n\pi x)\,dx = \int \psi^2(x)\cdot\frac{1}{2}\left[1-\cos(2n\pi x)\right]\,dx \to \frac{1}{2}\|\psi\|^2 \neq 0$$

In fact, if we set $g_n = f_n^2 = [\psi(x)]^2\sin^2(n\pi x)$, then $g_n \to \frac{1}{2}\psi^2(x)$ because

$$\int g_n(x)\varphi(x)\,dx = \int \psi^2(x)\sin^2(n\pi x)\varphi(x)\,dx$$
$$= \frac{1}{2}\int \psi^2\varphi\,dx - \frac{1}{2}\int \varphi^2\psi\cos(2\pi n x)\,dx$$
$$\to \frac{1}{2}\int \psi^2\varphi\,dx$$

So $g_n \rightharpoonup \frac{1}{2}\psi^2$ □

Proof. (b)

$$g_n = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

$\|g_n\| = \sqrt{n}$, (g_n) is unbounded, so $g_n \not\rightharpoonup g$. In fact, $g_n \rightharpoonup \delta \in \mathcal{D}'(\mathbb{R})$.

$$h_n = \begin{cases} n^{1/4} & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

$\|h_n\| = 0$, and (h_n) is strongly and weakly convergent to 0. $\frac{1}{2}$ is the critical value for L^2, and $\frac{1}{p}$ is the critical value for L^p. □

Remark 8.52.

If $f_n \rightharpoonup f$ and $f_n \to g$, then we must have $f = g$ because

$$\langle f_n, h\rangle \to \langle f, h\rangle \;\; \forall\, h \in \mathcal{H}$$
$$\langle f_n, h\rangle \to \langle g, h\rangle \;\; \forall\, h \in \mathcal{H}$$

Since $\langle f, h\rangle = \langle g, h\rangle \;\; \forall\, h$, we have that $f = g$.

8.7 The Banach-Alaoglu Theorem

Definition 8.53. ***Weakly Sequentially Compact***
page 208 and Notes 2/23/11

A set $K \subset \mathcal{H}$ is *weakly sequentially compact* if for any sequence $(x_n) \subset K$ there exists a subsequence (x_{n_k}) such that $x_{n_k} \rightharpoonup x \in K$.

Theorem 8.54. ***Banach-Alaoglu Theorem***
page 208 and Notes 2/23/11

Suppose that $\mathcal{H}$ is a separable Hilbert space and $\overline{B} = \{x \in \mathcal{H} \mid \|x\| \leq 1\}$ is the closed unit ball. Then $\overline{B}$ is weakly sequentially compact.

Remarks

1. $\overline{B}$ is not strongly compact if $\mathcal{H}$ is infinite-dimensional. Ex: $\{e_n\}$ is an orthonormal basis, but (e_n) has no convergent subsequence
2. This can be thought of as a replacement of the Heine-Borel theorem in the infinite-dimensional case

Proof. Let $\{y_k \mid k \in \mathbb{N}\}$ be a dense subset of $\mathcal{H}$. Consider $(\langle x_n, y_1 \rangle)_n \subset \mathbb{C}$. By Cauchy-Schwarz, $|\langle x_n, y\rangle \leq \|x_n\|\|y_1\| \leq \|y_1\|$, so the sequence is bounded, and thus there exists a subsequence of (x_n), denoted $(x_{n,1,k})_k = (x_{1,k})$ such that $\langle x_{1,k}, y_1\rangle$ converges as $k \to \infty$. Pick a subsequence $(x_{2,k})$ of $(x_{1,k})$ such that $\langle x_{2,k}, y_2\rangle$ converges as $k \to \infty$. Let $x_j = x_{j,j}$ be the diagonal sequence. Then $\langle x_j, y_n\rangle$ converges for every y_k as $j \to \infty$ in this dense subset of $\mathcal{H}$. This defines a bounded linear functional F on $D = \{y_k \mid k \in \mathbb{N}\}$. By the Bounded Linear Transformation Theorem, this extends to a bounded linear functional $\overline{F} : \mathcal{H} \to \mathbb{C}$ such that $\overline{F}(y_k) = \lim_{j\to\infty} \langle x_j, y_k\rangle$ for all $k \in \mathbb{N}$. By the Riesz Representation Theorem, there exists $x \in \mathcal{H}$ such that $\langle x, y_k\rangle = \lim_{j\to\infty} \langle x_j, y_k\rangle$ for all $k \in \mathbb{N}$. Since $\{y_k\}$ is dense in $\mathcal{H}$ and $\|x\| \leq 1$, $\langle x, y\rangle = \lim_{j\to\infty} \langle x_j, y\rangle$ for all $y \in \mathcal{H}$, and thus $x_j \rightharpoonup x$. $\|x\| \leq \liminf_{j\to\infty} \|x_j\| \leq 1$, so $x \in \overline{B}$. $\square$

Remark 8.55.
Notes 2/23/11

1. We don't need $\mathcal{H}$ to be separable (restrict to a closed subspace spanned by $\{x_n\}$ which is separable)
2. Generalization to Banach spaces: the unit ball of X^* is weak-$*$ compact (equivalent to being weak compact if X is reflexive, i.e. $X^{**} = X$)

Definition 8.56. ***Weakly Sequentially Closed***
Notes 2/23/11

A set $F \subset \mathcal{H}$ is *weakly sequentially closed* if whenever $(x_n) \subset F$ is a sequence and $x_n \rightharpoonup x$, then $x \in F$.

Example 8.57. ***Weakly Closed ⇒ Strongly Closed***
Notes 2/23/11

Weakly closed implies strongly closed, but not conversely if $\mathcal{H}$ is infinite-dimensional. For example, let

$$S = \{x \in \mathcal{H} \mid \|x\| = 1\}$$
$$\overline{B} = \{x \in \mathcal{H} \mid \|x\| \leq 1\}$$

S is not weakly closed because $(e_n) \subset S$, $e_n \rightharpoonup 0 \notin S$. $\overline{B}$ is weakly closed because if $x_n \rightharpoonup x$, then $\|x\| \leq \liminf \|x_n\|$. The weak closure of S is $\overline{B}$.

Definition 8.58. ***Weakly Sequentially Lower Semicontinuous***
page 208 and Notes 2/23/11

A function $f : D \subset \mathcal{H} \to \mathbb{R}$ is *weakly sequentially lower semicontinuous* if

$$x_n \rightharpoonup x \quad \Rightarrow \quad f(x) \leq \liminf_{n\to\infty} f(x_n)$$

Example: $\|\cdot\| : \mathcal{H} \to \mathbb{R}$ is weakly sequentially lower semicontinuous.

Remark 8.59.
Notes 2/23/11

Weakly sequentially lower semicontinuous implies strongly sequentially lower semicontinuous, but not conversely.

Theorem 8.60.
page 209 and Notes 2/23/11

Suppose that D is a weakly closed, bounded (in norm) subset in a Hilbert space $\mathcal{H}$ and $f : D \to \mathbb{R}$ is a weakly sequentially lower semicontinuous function. Then f is bounded from below ($m = \inf_{x\in D} f(x) > -\infty$) and there exists $x \in D$ such that $f(x) = m$.

8.8 Chapter Summary

We begin by defining what it means for a bounded linear operator P to be a *projection* (with "opposite" $Q = I - P$), and we explore relationship between projections and direct sum decompositions: P a projection $\Leftrightarrow X = \operatorname{ran} P \oplus \ker P$. We introduce *orthogonal projections* and show that they are bounded and self-adjoint. We explore the connection between orthogonal projections P ($\Rightarrow \mathcal{H} = \operatorname{ran} P \oplus \ker P$) and direct sum decompositions ($\mathcal{M}$ closed) $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ ($\Rightarrow P$, $\operatorname{ran} P = \mathcal{M}$, $\ker P = \mathcal{M}^\perp$).

Recall from Chapter 5 that a linear functional is bounded iff it is continuous. We introduce the *Riesz Representation Theorem*: for all $\varphi \in \mathcal{H}^*$, there exists $y \in \mathcal{H}$ such that $\varphi(x) = \langle y, x\rangle$. This gives us that all Hilbert spaces are self-dual: $\mathcal{H}^{**} = \mathcal{H}$. This is because the map $J_1 : \mathcal{H} \to \mathcal{H}^*$ defined by $J_1 y = \varphi_y$ identifies $\mathcal{H}$ with its dual space, $\mathcal{H}^*$. Similarly, we can define a map J_2 that identifies $\mathcal{H}^*$ with its dual space, $\mathcal{H}^{**}$.

Thus, $\mathcal{H}$ and $\mathcal{H}^{**}$ (and $\mathcal{H}^*$) have the same cardinality. And since we know (Chapter 5) that for every $x \in \mathcal{H}$ we can define a functional $F_x \in \mathcal{H}^{**}$ by $F_x(\varphi) = \varphi(x)$, we therefore know that all linear functionals in $\mathcal{H}^{**}$ are of this form.

We use the Riesz Representation Theorem to prove the existence of the *adjoint* of a bounded operator on a Hilbert space: $\langle x, Ay\rangle = \langle A^*x, y\rangle$. Examples:

- Matrix: $A^* = A^T$ ($\overline{A^T}$ if A is complex)
  - $\langle x, Ay\rangle = x^T Ay,\ \langle A^*x, y\rangle = (A^*x)^T y = x^T(A^*)^T y$
- Integral operator $Kf(x) = \int_0^1 k(x,y)f(y)\,dy$: $K^*f(x) = \int_0^1 \overline{k(y,x)}f(y)\,dy$
- Shift operators: $S^* = T,\ T^* = S$

We verify that for a bounded linear operator A, a solvability condition for $Ax = y$ is that $\langle y, z\rangle = 0$ for all $z \in \ker A^* \;\Leftrightarrow\; \operatorname{ran} A \subset (\ker A^*)^\perp$. We use this fact to prove that for a bounded linear operator A,

$$\overline{\operatorname{ran} A} = (\ker A^*)^\perp, \qquad \ker A = (\operatorname{ran} A^*)^\perp.$$

Equivalently,

$$\mathcal{H} = \underbrace{(\ker A^*)^\perp}_{\overline{\operatorname{ran} A}} \oplus \underbrace{(\operatorname{ran} A)^\perp}_{\ker A^*}.$$

Next we have some definitions. We define what it means for a bounded linear operator to be *self-adjoint*, and we prove that for a bounded self-adjoint operator A,

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle|, \qquad \|A^*A\| = \|A\|^2.$$

Examples:

- A matrix is self-adjoint if it is symmetric (or Hermitian, if it is complex).
- An integral operator $Kf(x) = \int_0^1 k(x,y)f(y)\,dy$ is self-adjoint if $k(x,y) = \overline{k(y,x)}$

We say that an operator is *unitary/orthogonal* if it is invertible and $\langle Ux, Uy\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} \;\Leftrightarrow\; U^*U = UU^* = I$. We say that an operator is *normal* if $T^*T = TT^*$. (Self-adjoing and unitary operators are normal.)

Now we revisit *weak convergence*. For Hilbert spaces, the Riesz Representation Theorem gives us an equivalent definition: $x_n \rightharpoonup x$ if $\langle x_n, y\rangle \to \langle x, y\rangle\ \forall\ y \in \mathcal{H} \;\Leftrightarrow\; \varphi(x_n) \to \varphi(x)\ \forall\ \varphi \in \mathcal{H}^*$. We mention 3 reasons why a sequence may converge weakly but not strongly: *oscillation*, *concentration*, and *escape to infinity*. We prove that for a weakly convergent sequence (x_n), $\|x\| \le \liminf \|x_n\|$. We also prove that if $\lim \|x_n\| = \|x\|$, then (x_n) converges to x strongly. The *Banach-Alaoglu Theorem* tells us that the closed unit ball of a Hilbert space is weakly compact.

We define what it means for a function to be *convex*, and we say a few words about *lower semicontinuous* functions. We finish the chapter with *Mazur's Theorem*, which tells us that if $x_n \rightharpoonup x$, then there exists a sequence (y_n) of finite convex combinations of $\{x_n\}$ that converges strongly to x.

9 The Spectrum of Bounded Linear Operators

9.0 Introduction

Remark 9.1.
page 215 and Notes 3/2/11

Consider the following initial boundary value problem for a variable coefficient, linear equation:

$$\begin{aligned} &u_t = u_{xx} - q(x)u && 0 < x < 1,\ t > 0, \\ &u(0,t) = 0,\ u(1,t) = 0 && t \geq 0, \\ &u(x,0) = f(x) && 0 \leq x \leq 1 \end{aligned}$$

Using separation of variables, we assume

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) u_n(x)$$

where $\{u_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of $L^2([0,1])$. We find that

$$\frac{da_n}{dt} = -\lambda_n a_n n$$

and the u_n satisfy

$$-\frac{d^2 u_n}{dx^2} + q u_n = \lambda_n u_n$$

Then the u_n are eigenvectors of the linear operator A. Thus, $Au_n = \lambda_n u_n$, where A is defined by

$$Au = -\frac{d^2 u}{dx^2} + qu$$

We want a complete set of eigenvectors of A, or equivalently, to diagonalize A. This is an example of what we do in spectral theory.

9.1 Diagonalization of Matrices

Remark 9.2.
page 218 and Notes 3/2/11

The concept of the spectrum of an operator on a Banach/Hilbert space is a generalization of eigenvalues for matrices. Let $A \in \mathcal{B}(X)$. When $\dim X < \infty$ then we can identify it with a a matrix $\tilde{A}$. For any $\lambda \in \mathbb{C}$ we have two possibilities:

1. $\lambda I - A$ is nonsingular $\Leftrightarrow \det(\lambda I - A) = 0 \Leftrightarrow (\lambda I - A)^{-1}$ exists
2. $\lambda I - A$ is singular $\Leftrightarrow$ there exists x_0 such that $(\lambda I - A)x_0 = 0$. Thus, $Ax_0 = \lambda x_0$, λ is an eigenvalue, and x_0 is an eigenvector.

What happens if $\dim X = \infty$???

9.2 The Spectrum

Definition 9.3. ***Resolvent Set***
page 218 and Notes 3/2/11

The *resolvent set* of a bounded operator A on a Banach space X is the set

$$\begin{aligned} \rho(A) &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is invertible } \} \\ \text{(by the bounded inverse theorem)} \qquad &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \in \mathcal{B}(X)\} \\ &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is 1-1 and onto } \} \end{aligned}$$

Definition 9.4. ***Spectrum***
page 218 and Notes 3/2/11

The *spectrum* of A is the set

$$\begin{aligned} \sigma(A) &= \mathbb{C} \setminus \rho(A) \\ &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is not invertible } \} \end{aligned}$$

Definition 9.5. ***Point Spectrum, Continuous Spectrum, Residual Spectrum***
page 219 and Notes 3/2/11

In general, $\sigma(A)$ can be expressed as $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$, where

1. $\sigma_p(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is not 1-1 } \}$
   $\sigma_p(A)$ is called the *point spectrum* of A. In this case, since $(\lambda I - A)$ is not 1-1, there exists $x_0 \in \ker(\lambda I - A)$ such that $(\lambda I - A)x_0 = 0 \Leftrightarrow Ax_0 = \lambda x_0$
2. $\sigma_c(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is 1-1 but not onto and } \overline{\operatorname{ran}(\lambda I - A)} = X\}$
   $\sigma_c(A)$ is called the *continuous spectrum* of A
3. $\sigma_r(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is 1-1 but not onto and } \overline{\operatorname{ran}(\lambda I - A)} \neq X\}$
   $\sigma_r(A)$ is called the *residual spectrum* of A

Example 9.6. ***Point, Continuous, and Residual Spectra Examples***
Notes 3/7/11

1. A matrix on $\mathbb{C}^n$ has pure point spectrum
2. $M : L^2([0,1]) \to L^2([0,1])$, $f \mapsto xf$, $\sigma(M) = [0,1]$ has pure continuous spectrum
3. Consider the right shift operator S on $\ell^2(\mathbb{N})$. $\lambda = 0$ is in the residual spectrum

Example 9.7.
Notes 3/2/11

Consider the Banach space $X = C([0,1])$ with the $\| \cdot \|_\infty$ norm. Define $A : X \to X$ by $Af(x) = xf(x)$. The boundedness of A follows exactly as in HW7 (even though $X = L^2([0,1])$ on the HW, since we can take $\sup x = 1$). Find $\sigma(A)$. Claim: $\sigma(A) = \sigma_r(A) = [0,1]$.

For any $\lambda \in \mathbb{C}, f \in C([0,1])$, we have

$$(\lambda I - A)f(x) = (\lambda - x)f(x) = 0$$

If $\lambda \neq x$ then $f(x) = 0$. If $\lambda \notin [0,1]$ then $\sigma_p = \emptyset$.
For all $\lambda \notin [0,1]$, is $(\lambda I - A)$ onto? For every $g \in C([0,1])$, we want f such that $f(x)(\lambda - x) = g(x) \Rightarrow f(x) = \frac{g(x)}{\lambda - x} \in C([0,1])$, since $\lambda \notin [0,1]$ implies that $\lambda - x \neq 0 \ \forall \ x \in [0,1]$. Thus, $(\lambda I - A)$ is onto, and we can conclude that $\sigma(A) \subseteq [0,1]$.
It will be enough to prove the claim to show that $[0,1] \subseteq \sigma_r(A)$. Why? $[0,1] \subseteq \sigma_r(A) \subseteq \sigma(A) \subseteq [0,1]$.
Pick $\lambda \in [0,1]$. For every $g \in \operatorname{ran}(\lambda I - A)$ we have that

$$g(x) = (\lambda - x)f(x) \text{ for some } f \in X = C([0,1])$$
$$g(\lambda) = 0$$

So $h(x) = 1 \notin \operatorname{ran}(\lambda I - A)$, since $g(\lambda) = 0 \neq 1$. Therefore $(\lambda I - A)$ is not onto.

If $h \in \overline{\operatorname{ran}(\lambda I - A)}$ then there exists $(g_n) \subset \operatorname{ran}(\lambda I - A)$ such that $g_n \to h$. $h(\lambda = \lim_{n\to\infty} g_n(\lambda)(\lambda I - A) = 0$. Thus, $\mathbf{1} \notin \overline{\operatorname{ran}(\lambda I - A)}$, so $\lambda \in \sigma_r(A)$.

Example 9.8.
page 219

Example 9.5 on page 219

Definition 9.9. ***Resolvent***
page 220 and Notes 3/4/11

For $\lambda \in \rho(A)$, we define the *resolvent* of A at λ to be

$$R_\lambda = (\lambda I - A)^{-1}, \qquad R_\lambda : \rho(A) \subset \mathbb{C} \to \mathcal{B}(\mathcal{H})$$

Example 9.10. ***Neumann Series***
page 220 and Notes 3/4/11

If $\|A\| < 1$ then $(I - A)$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \ldots$$

To show this, we define the partial sum:

$$S_N = I + A + A^2 + \ldots + A^N$$

Next, we show that the sequence of partial sums is Cauchy:

$$\|A^{M+1} + \ldots + A^N\| \leq \|A^{M+1}\| + \ldots + \|A^N\| \leq \|A\|^{M+1} + \ldots + \|A\|^N$$
$$\leq \sum_{n=M+1}^{N} \|A\|^n$$

$\sum_{n=1}^{\infty} < \infty$ if $\|A\| < 1$, so the partial sums are Cauchy. Thus, $\sum_{n=0}^{\infty} A^n$ is Cauchy in $\mathcal{B}(\mathcal{H})$, and it converges since $\mathcal{B}(\mathcal{H})$ is complete.

(See Remark 9.12.)

Example 9.11.
Notes 3/4/11

1. If $|\lambda| > \|A\|$ then $\lambda \in \rho(A)$
   $(\lambda I - A)^{-1} = \left[\lambda\left(I - \frac{A}{\lambda}\right)\right]^{-1} = \frac{1}{\lambda}\left(I - \frac{A}{\lambda}\right)^{-1}$
   $\uparrow$ this exists if $\|A/\lambda\| < 1 \Rightarrow \|A\| < |\lambda|$
2. The resolvent set $\rho(A)$ is open in $\mathbb{C}$
   Suppose $\lambda_0 \in \rho(A)$. We write:

$$(\lambda I - A) = \lambda_0 I - A + (\lambda - \lambda_0)I = (\lambda_0 I - A)\left[I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1}\right]$$
$$(\lambda I - A)^{-1} = \left[I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1}\right]^{-1}(\lambda_0 I - A)^{-1}$$
$$\uparrow \text{exists if } |\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 I - A)^{-1}\|}$$

3. $R_\lambda : \lambda \mapsto (\lambda I - A)^{-1}$
   R_λ is an operator-valued analytic function on the open set $\rho(A) \subset \mathbb{C}$
4. $\sigma(A) \neq \emptyset$

Remark 9.12.
Notes 3/4/11

In Example 9.10, it is not necessary that $\|A\| < 1$ for $(I - A)^{-1} = I + A + A^2 + \ldots$ to converge. Rather, we require that $\lim_{n\to\infty} \|A^n\|^{1/n} < 1$.

Definition 9.13. ***Spectral Radius***
page 220 and Notes 3/4/11

$r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$ is the *spectral redius* of A. This is the radius of the smallest disc in $\mathbb{C}$ centered at 0 that contains $\sigma(A)$. Also, $r(A) \leq \|A\|$.

Theorem 9.14.
page 220 and Notes 3/4/11

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} \quad \text{(and the limit exists)}$$

Proof. Let $a_n = \log \|A^n\|$. (If $\|A^n\| = 0$ for some n, i.e. A is *nilpotent*, then $r(A) = 0$.) Then

$$\begin{aligned} a_{m+n} &= \log \|A^{m+n}\| \\ &\leq \log \|A^n\| + \log \|A^n\| \\ &\leq a_m + a_n \quad \text{(subadditive)} \end{aligned}$$

We want to show that $\lim_{n\to\infty} \frac{a_n}{n}$ exists, where $\frac{a_n}{n} = \log \|A^n\|^{1/n}$. Fix n, m and write $n = mp + q$ with $0 \leq q < m$. Then we have

$$\begin{aligned} a_n = a_{mp+q} &\leq a_{mp} + a_q \\ \frac{a_n}{n} &\leq \frac{a_{mp}}{n} + \frac{a_q}{n} \end{aligned}$$

Note that $a_{mp} \leq p a_m$. Let $n \to \infty$ with m fixed. Then $\frac{p}{n} \to \frac{1}{m}$ as $n \to \infty$, and

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m} \tag{9.1}$$

Taking the limit of (9.1) as $m \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \liminf_{m\to\infty} \frac{a_m}{m}$$

So $\limsup_{n\to\infty} \frac{a_n}{n} = \limsup_{n\to\infty} \frac{a_n}{n}$, and the sequence converges. □

Example 9.15. ***Example for Theorem 9.14***
Notes 3/4/11

$$\begin{aligned} A &= \mu I \\ \lambda I - A &= (\lambda - \mu) I \\ \sigma(A) &= \mu \end{aligned}$$

$$\begin{aligned} \|A\| &= |\mu| = r(A) \\ \|A^n\|^{1/n} &= |\mu| \end{aligned}$$

Corollary 9.16.
page 221 and Notes 3/4/11

If A is self-adjoint then $r(A) = \|A\|$.

Proof. $\|A^2\| = \|A\|^2$ and $\|A^{2^n}\| = \|A\|^{2^n}$, so $\liminf_{n\to\infty} \|A^n\|^{1/n} = \|A\|$ by taking the subsequence $n = 2^m$. □

9.3 The Spectral Theorem for Compact, Self-Adjoint Operators

9.3.1 Bounded, Self-Adjoint Operators

Theorem 9.17.
page 222 and Notes 3/7/11

If A is bounded and self-adjoint, then every eigenvalue of A is real and eigenvectors with different eigenvalues are orthogonal.

Related to Theorem 9.21.

Proof. If $Ax = \lambda x$, then

$$\begin{aligned}\langle x, Ax\rangle &= \langle x, \lambda x\rangle = \lambda\|x\|^2 \\ \langle Ax, x\rangle &= \langle \lambda x, x\rangle = \overline{\lambda}\|x\|^2\end{aligned}$$

If A is self-adjoint (and $x \neq 0$), then $\lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

Case: A has pure point spectrum.
If $Ax = \lambda x$ and $Ay = \mu y$, $x, y \neq 0$, $\lambda \neq \mu$, then

$$\left.\begin{aligned}\langle x, Ay\rangle &= \mu\langle x, y\rangle \\ \langle Ax, y\rangle &= \overline{\lambda}\langle x, y\rangle = \lambda\langle x, y\rangle\end{aligned}\right\} A = A^*, \text{ so } \mu\langle x, y\rangle = \lambda\langle x, y\rangle$$

If $\lambda \neq \mu$, then $\langle x, y\rangle = 0$, i.e. $x \perp y$.

What about the continuous and residual spectra?

$$\begin{aligned}\|(A - \lambda I)x\|^2 &= \langle (A - aI)x - ibx, (A - aI)x - ibx\rangle \qquad \text{where } \lambda = a + ib \\ &= \langle (A - aI)x, (A - aI)x\rangle + \cancel{\langle -ibx, (A - aI)x\rangle} + \cancel{\langle (A - aI)x, -ibx\rangle} + \langle -ibx, -ibx\rangle \\ &= \|(A - aI)x\|^2 + b^2\|x\|^2 \\ &\geq b^2\|x\|^2\end{aligned}$$

Continuous Spectrum: See Proposition 9.18 and Remark 9.19.
Residual Spectrum: See Proposition 9.20. □

Proposition 9.18.
page 223 and Notes 3/7/11

$$|\text{Im }\lambda| \cdot \|x\| \leq \|(A - aI)x\|$$

Remark 9.19.
Notes 3/7/11

Proposition 9.18 says that if $(A - \lambda I)x = y$, then $|\text{Im } \lambda| \cdot \|x\| \le \|y\|$. This means that if $\lambda \in \mathbb{R}$, we can estimate the solution, x, in terms of the RHS, y.

Applying this to the proof of Theorem 9.17, we see that if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, it follows that

(a) $(A - \lambda I)$ is 1-1 because if $(A - \lambda I)x = 0$ then $|\text{Im } \lambda| \|x\| = 0 \Rightarrow x = 0$.

(b) $(A - \lambda I)$ has closed range. If $y_n = (A - \lambda I)x_n, y_n \in \text{ran}(A - \lambda I), y_n \to y$, then we can bound

$$\underbrace{\|x_m - x_n\|}_{\therefore \text{ Cauchy}} \le C \underbrace{\|y_m - y_n\|}_{\text{Cauchy}}$$

So $x_n \to x$, $(A - \lambda I)x = y$, and $y \in \text{ran}(A - \lambda I)$. So if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $(A - \lambda I)$ is 1-1 with closed range, so there is no complex-valued continuous spectrum.

Proposition 9.20.
page 224 and Notes 3/7/11

If A is bounded and self-adjoint, then the residual spectrum is empty.

Proof. If λ is in the residual spectrum, then there exists $y \in \mathcal{H}$ such that $\langle (A - \lambda I)x, y \rangle = 0 \ \forall \ x \in \mathcal{H}$, so $y \perp \overline{\text{ran}(A - \lambda I)}$, $y \neq 0$. Since A is self-adjoint, $\left\langle x, (A - \overline{\lambda} I)y \right\rangle = 0 \ \forall \ x \in \mathcal{H}$. This implies that $(A - \overline{\lambda} I)y = 0$, so y is an eigenvector of A with eigenvalue $\overline{\lambda}$. We have 2 cases:

1. $\lambda \in \mathbb{C} \setminus \mathbb{R} \Rightarrow$ impossible (A has real eigenvalues)
2. $\lambda \in \mathbb{R}$. Then λ is in the point and residual spectra $\Rightarrow$ impossible.

□

Theorem 9.21.
page 223 and Notes 3/7/11

If A is a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$, then $\sigma(A)$ is real and contained in the interval $[-\|A\|, \|A\|]$. The residual spectrum is empty.

Related to Theorem 9.17.

Proposition 9.22.
page 223

If A is a bounded operator on a Hilbert space (not necessarily self-adjoint!) and $\lambda \in \sigma_r(A)$, then $\overline{\lambda} \in \sigma_p(A^*)$. In other words, $\sigma_r(A) \subseteq \sigma_p(A^*)$.

Remark 9.23. ***Bounded, Self-Adjoint Operators***

Bounded, self-adjoint operators have

- Spectral radius $r(A) = \|A\|$ (See Corollary 9.16)
- Real eigenvalues (See Theorem 9.17)
- Orthogonal eigenvectors (See Theorem 9.17)
- Empty residual spectrum (See Proposition 9.20)

9.3.2 Compact Operators

Definition 9.24. ***Compact Operator***
Notes 3/9/11

$K : \mathcal{H} \to \mathcal{H}$, $D \in \mathcal{B}(\mathcal{H})$ is *compact* if it maps bounded sets to precompact sets.

Remark 9.25. ***Precompact***
Notes 3/9/11

Remember: a set is *precompact* if it is bounded and "almost" finite-dimensional.

Example 9.26. ***The Hilbert Cube***
page 230 and Notes 3/9/11

Let $\mathcal{H} = \ell^2(\mathbb{N})$. The *Hilbert cube*

$$C = \left\{ (x_1, x_2, \ldots, x_n, \ldots) \;\middle|\; |x_n| \leq \frac{1}{n} \right\}$$

is closed and precompact. Hence, C is a compact subset of $\mathcal{H}$.

Example 9.27. ***Diagonal Operators Are Compact***
page 230

The diagonal operator : $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$A(x_1, x_2, \ldots, x_n, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots)$$

is compact iff $\lambda_n \to 0$ as $n \to \infty$.

Example 9.28. ***Compactness of Operators***
Notes 3/9/11

1. Any operator with finite rank (rank A = dim ran A) is compact
2. $I : \mathcal{H} \to \mathcal{H}$ is not compact if dim $\mathcal{H} = \infty$
3. $L^2([0,1])$, $Kf(x) = \int_0^x f(y)\,dy$ is a compact operator. If $\|f\|_{L^2} \le M$, then

$$\left|\int_0^x f(y)\,dy\right| \le \int_0^1 |f(y)|\,dy \le \left(\int_0^1 |f(y)|^2\,dy\right)^{1/2} \le M$$

Define $F(x) = \int_0^x f(y)\,dy$. Then

$$|F(x_2) - F(x_1)| = \left|\int_{x_1}^{x_2} f(y)\,dy\right| \le \left(\int_{x_1}^{x_2} 1 \cdot dy\right)^{1/2} \left(\int_{x_1}^{x_2} |f(y)|^2\,dy\right)^{1/2} \le M|x_2 - x_1|^{1/2}$$

$\{Kf \mid \|f\| \le M\}$ is bounded and equicontinuous. Thus, $H^2([0,1])$ is compactly embedded in $L^2([0,1])$. It follows that $\{Kf \mid \|f\|_{L^2} \le M\}$ is precompact in $C([0,1])$ by Arzela-Ascoli, so it is precompact in $L^2([0,1])$.

If $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$, then $Kf(x) = \sum_{n=1}^{\infty} \frac{b_n}{n\pi} - \sum_{n=1}^{\infty} \frac{b_n}{n\pi}\cos(n\pi x)$

9.3.3 Compact, Self-Adjoint Operators

Remark 9.29.
page 223 and Notes 3/9/11

Given: $A : \mathcal{H} \to \mathcal{H}$, A is compact and self-adjoint, $\mathcal{H}$ is a separable Hilbert space
We will prove:

1. A has at least one eigenvalue
2. If A leaves a subspace $M \subset \mathcal{H}$ *invariant* ($A : M \to M$), then A leaves $M^\perp$ invariant, and $\mathcal{H} = M \oplus M^\perp$

Idea: if we have $A\varphi_n = \lambda_n \varphi_n$, then we can get the largest eigenvalue by maximizing $A(\sum c_n \varphi_n) = \sum \lambda_n c_n \varphi_n$.

Theorem 9.30.
page 225 and Notes 3/9/11

Suppose $A : \mathcal{H} \to \mathcal{H}$ is compact and self-adjoint. Then A has an eigenvector with eigenvalue λ with $\lambda = \|A\|$ and/or $\lambda = -\|A\|$.

Proof. Recall: since A is self-adjoint, $\|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle|$. Choose a sequence $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1$

and $\langle x_n, Ax_n\rangle \to \lambda$ as $n \to \infty$, $\lambda = \pm\|A\|$. Then we have

$$\begin{aligned}
\|(A-\lambda I)x_n\|^2 &= \langle (A-\lambda I)x_n, (A-\lambda I)x_n\rangle \\
&= \langle Ax_n, Ax_n\rangle - 2\lambda\langle x_n, Ax_n\rangle + \lambda^2\langle x_n, x_n\rangle \\
&= \underbrace{\|Ax_n\|^2}_{\le\|A\|^2\|x_n\|^2=\lambda^2} - 2\lambda\langle x_n, Ax_n\rangle + \lambda^2 \\
&\le 2\lambda^2 - 2\lambda\langle x_n, Ax_n\rangle \qquad \to 0 \text{ as } n\to\infty
\end{aligned}$$

So $(A-\lambda I)x_n \to 0$ as $n \to \infty$, and thus $x_n - \frac{1}{\lambda}Ax_n \to 0$ (assuming $\lambda \neq 0$, in which case $\|A\| = 0$ and everything is an eigenvalue). Since (x_n) is bounded ($\|x_n\| = 1\ \forall\ n$), $Ax_n \to y$ by the compactness of A. So $x_n \to \frac{y}{\lambda}$ and $(A-\lambda I)y = 0$. $\|y\| = \lambda \neq 0$, since $\|x_n\| = 1$ and $x_n \to y$. So A has eigenvector y with eigenvalue λ. □

Proposition 9.31.
page 224 and Notes 3/9/11

1. Any nonzero eigenvalue of a compact, self-adjoint operator has a finite *multiplicity* (multiplicity $\equiv$ the dimension of the eigenspace).
2. If λ_n is a sequence of eigenvalues and $\lambda_n \to L$, then we must have that $L = 0$.

Theorem 9.32. ***Spectral Theorem for Compact, Self-Adjoint Operators***
page 225 and Notes 3/11/11

If $A : \mathcal{H} \to \mathcal{H}$ is a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$ then there is a finite or countably infinite sequence (λ_n) of nonzero real eigenvalues and orthogonal eigenvectors (φ_n) such that

$$A\varphi_n = \lambda_n\varphi_n$$

where $|\lambda_1| \ge |\lambda_2| \ge \ldots$. $\lambda_n \to 0$ as $n \to \infty$ if there are infinitely many λ_n's and

$$\begin{aligned}
Ax &= \sum_n \lambda_n\langle\varphi_n, x\rangle\varphi_n \\
x &= \sum\langle\varphi_n, x\rangle\varphi_n + n \qquad \text{where } n\in\ker A, \quad \ker A \perp \underbrace{<\varphi_n>}_{\text{span}}
\end{aligned}$$

Let $P_n : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto the eigenspace with eigenvalue λ_n (eigenvectors of bounded, self-adjoint operators are orthogonal; see Theorem 9.17). Then

$$A = \sum \lambda_n P_n$$

We are representing A as a sum of linear projections because $\lambda_n \to 0$, and so the sum converges uniformly.

Proof. To see that the sum converges uniformly to A, we compute

$$\|Ax - \sum_{n=1}^{N}\lambda_n P_n x\| = \sum_{n=N+1}^{\infty}|\lambda_n\langle\varphi_n, x\rangle\varphi_n|^2 \le |\lambda_{N+1}|^2\|x\|^2$$

Also, if we let P_0 be the orthogonal projection onto $\ker A$, then

$$P_0 + \sum P_n = I$$

is strongly convergent. This is an example of what's called "resolution of the identity." Note that the λ_i's gave us uniform convergence above. For bounded (and unbounded) self-adjoint operators with continuous spectrum we need to use resolutions of identity that involve integrals (instead of sums). □

9.4 Functions of Operators = Functional Calculus

Definition 9.33. ***Function of an Operator***
page 232 and Notes 3/11/11

If $f : \sigma(A) \subset \mathbb{C} \to \mathbb{C}$ is a bounded function, then we define

$$f(A) = \sum f(\lambda_n) P_n + f(0) P_0$$

- f is uniformly convergent if $f(\lambda_n) \to 0$ as $n \to \infty$
- f is strongly convergent if $f(\lambda_n) \not\to 0$ as $n \to \infty$

Note that $\sigma(A) = \{\lambda_n\} \cup \{0\}$ if $\dim H = \infty$

- If there are finitely many λ_n, then $0 \in \sigma_p(A)$
- If there are countably many λ_n, then $0 \in \sigma_c(A)$

Example 9.34.
Notes 3/11/11

Suppose A is a positive (see Definition 8.32), self-adjoint compact operator. Then

$$\langle x, Ax \rangle \geq 0 \quad \text{implies} \quad \lambda_n \geq 0 \; \forall \; n$$

We can define the positive square root of A as

$$\sqrt{A} = \sum \lambda_n^{1/2} P_n$$

$$\left(\sqrt{A}\right)^2 = \sum \lambda_n P_n = A$$

In general, if A is compact then

$$T = A^*A \quad \text{is positive and self-adjoint because} \quad \langle x, Tx \rangle = \langle x, A^*Ax \rangle = \langle Ax, Ax \rangle \geq 0$$

$$\sqrt{T} = |A|, \qquad |A|^2 = T = A^*A$$

Definition 9.35. ***Polar Decomposition***
page 217 and Notes 3/11/11

$A = U|A|$, where $U : \operatorname{ran}|A| \to \operatorname{Im} A$ is a unitary operator

Definition 9.36. ***Fredholm Operator, Index***
Notes 3/11/11

A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is *Fredholm* if

(a) $\operatorname{ran} A$ is closed
(b) $\dim \ker A$ is finite
(c) $\operatorname{codim} \operatorname{ran} A$ is finite $\Leftrightarrow$ $\dim \ker A^*$ is finite
 - $\operatorname{codim} \operatorname{ran} A = \dim \ker A^*$ (recall that $\mathcal{H} = \operatorname{ran} A \oplus \ker A^*$ when $\operatorname{ran} A$ is closed)

We define the *index* by

$$\text{index } A = \dim(\ker A) - \operatorname{co}\dim(\operatorname{ran} A) = \dim(\ker A) - \dim(\ker A^*)$$

Example 9.37. ***Fredholm or not?***
Notes 3/11/11

(a) I is Fredholm with index $= 0$
(b) $A(x_1, x_2, x_3, \ldots) \mapsto (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$ is not Fredholm because the range is not closed
(c) The right shift operator, S, is Fredholm with index $= -1$

If A is Fredholm with $\text{index}(A) = 0$ then we have Fredholm alternative for solving the equation $Ax = y$, and there are 2 possibilities:

1. A is one-to-one and we can solve the equation for every $y \in \mathcal{H}$
2. A is not one-to-one, and we can only solve the equation if $y \perp \ker A^*$

Theorem 9.38. ***Riesz-Schauder Theorem***
Notes 3/11/11

If K is a compact, self-adjoint operator and $\lambda \neq 0$ then $A = \lambda I - K$ is Fredholm with index 0.

9.5 Chapter Summary

$$\begin{aligned}
U^*AU &= U^*(AU) = U^*\left(A\begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix}\right) \\
&= U^*\begin{bmatrix} Au_1 & Au_2 & \cdots & Au_k \end{bmatrix} \\
&= U^*\begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_k u_k \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \cdots & \lambda_k e_k \end{bmatrix} \qquad \text{(because } U^* u_k = e_k\text{)} \\
&= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} = D
\end{aligned}$$

Operator	Spectrum	Point	Continuous	Residual
Bounded, Linear	Closed & Nonempty, $r(A) = \lim \|A^n\|^{1/n}$			$\lambda \in \sigma_r(A) \Rightarrow \overline{\lambda} \in \sigma_p(A^*)$
Bounded, Self-Adjoint	$\sigma(A) \subset [-\|A\|, \|A\|]$ $r(A) = \|A\|$	real	real	empty
Compact, Self-Adjoint		$-\|A\| \in \sigma_p(A)$ or $\|A\| \in \sigma_p(A)$	$\sigma_c(A) = \{0\}$ or $\sigma_c(A) = \emptyset$	empty