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Professor: Shkoller
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Author: Jeff Irion
http://www.math.ucdavis.edu/~jlirion

Contents

1 Measure Theory

Theorem 1.1. ***Fubini's Theorem***
http://en.wikipedia.org/wiki/Fubini%27s_theorem

Suppose A and B are complete measure spaces. Suppose $f(x, y)$ is $A \times B$ measurable. If

$$\int_{A\times B} |f(x, y)|\, d(x, y) < \infty$$

where the integral is taken with respect to a product measure on the space over $A \times B$, then

$$\int_A \left(\int_B f(x, y)\, dy\right)\, dx = \int_B \left(\int_A f(x, y)\, dx\right)\, dy = \int_{A\times B} f(x, y)\, d(x, y)$$

the first two integrals being iterated integrals with respect to two measures, respectively, and the third being an integral with respect to a product of these two measures.

Corollary:
If $f(x, y) = g(x)h(y)$ for some functions g and h, then

$$\int_A g(x)\, dx \int_B h(y)\, dy = \int_{A\times B} f(x, y)\, d(x, y)$$

the third integral being with respect to a product measure.

Theorem 1.2. ***Tonelli's Theorem***
http://en.wikipedia.org/wiki/Fubini%27s_theorem#Tonelli.27s_theorem

Suppose that A and B are σ-finite measure spaces, not necessarily complete. If either

$$\int_A \left(\int_B |f(x, y)|\, dy\right)\, dx < \infty \text{ or } \int_B \left(\int_A |f(x, y)|\, dx\right)\, dy < \infty$$

then

$$\int_{A\times B} |f(x, y)|\, d(x, y) < \infty$$

and

$$\int_A \left(\int_B f(x, y)\, dy\right)\, dx = \int_B \left(\int_A f(x, y)\, dx\right)\, dy = \int_{A\times B} f(x, y)\, d(x, y)$$

Remark 1.3. ***Fubini vs. Tonelli***
http://en.wikipedia.org/wiki/Fubini%27s_theorem

Tonelli's theorem is a successor of Fubini's theorem. The conclusion of Tonelli's theorem is identical to that of Fubini's theorem, but the assumptions are different. Tonelli's theorem states that on the product of two -finite measure spaces, a product measure integral can be evaluated by way of an iterated integral for nonnegative measurable functions, regardless of whether they have finite integral. A formal statement of Tonelli's theorem is identical to that of Fubini's theorem, except that the requirements are now that (X, A, μ) and (Y, B, ν) are σ-finite measure spaces, while f maps $X \times Y$ to $[0, \infty]$.

Theorem 1.4. ***Cauchy-Schwarz Inequality***
http://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality

Formal Statement: For all vectors x, y of an inner product space,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$$
$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Square of a Sum:

$$\left|\sum_{i=1}^{n} x_i y_i\right|^2 \leq \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2$$

In L^2:

$$\left|\int f(x)g(x)\,dx\right|^2 \leq \int |f(x)|^2\,dx \int |g(x)|^2\,dx$$

Theorem 1.5. ***Hölder's Inequality***
Theorem 12.54 on page 356

Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, then $fg \in L^1(X, \mu)$ and

$$\underbrace{\left|\int fg\,d\mu\right|}_{\|fg\|_1} \leq \|f\|_p \|g\|_q$$

Note: The Cauchy-Schwartz Inequality is a special case of Hölder's Inequality for $p = q = 2$.

Theorem 1.6. ***Minkowski's Inequality***
201A Notes 11/3/10

$$\|x + y\|_p \leq \|x\|_p + \|y\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Theorem 1.7. ***Young's Inequality***
Theorem 12.58 on page 359

Let $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q$$

Theorem 1.8. ***Lebesgue Dominated Convergence Theorem***
Theorem 12.35 on page 348

Suppose that (f_n) is a sequence of integrable functions, $f_n : X \to \overline{\mathbb{R}}$, on a measure space $(X, \mathcal{A}, \mu)$ that converges pointwise to a limiting function $f : X \to \overline{\mathbb{R}}$. If there is an integrable function $g : X \to [0, \infty]$ such that

$$|f_n(x)| \le g(x) \quad \forall\ x \in X,\ n \in \mathbb{N}$$

then f is integrable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Theorem 1.9. ***Monotone Convergence Theorem***
Theorem 12.33 on page 347

Suppose that (f_n) is a monotone increasing sequence of nonnegative, measurable functions $f_n : X \to [0, \infty]$ on a measurable space $(X, \mathcal{A}, \mu)$. Let $f : X \to [0, \infty]$ be the pointwise limit, i.e.

$$\lim_{n\to\infty} f_n(x) = f(x)$$

Then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Lemma 1.10. ***Fatou's Lemma***
Theorem 12.34 on page 347

If (f_n) is any sequence of nonnegative measurable functions $f_n : X \to [0, \infty]$ on a measure space $(X, \mathcal{A}, \mu)$, then

$$\int \left(\liminf_{n\to\infty} f_n \right) d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu$$

Equivalently,

$$\limsup_{n\to\infty} \int f_n \, d\mu \le \int \left(\limsup_{n\to\infty} f_n \right) d\mu$$

Theorem 1.11. ***Lebesgue Differentiation Theorem***
`http://en.wikipedia.org/wiki/Lebesgue_differentiation_theorem`

For a Lebesgue integrable function f on $\mathbb{R}^n$, the indefinite integral is a set function which maps a measurable set A to the Lebesgue integral of $f \cdot \mathbf{1}_A$, written as:

$$\int_A f \, d\lambda$$

The derivative of this integral at x is defined to be

$$\lim_{B \to x} \frac{1}{|B|} \int_B f \, d\lambda$$

where $|B|$ denotes the volume of a ball centered at x, and $B \to x$ means that the radius of the ball is going to zero. The Lebesgue differentiation theorem states that this derivative exists and is equal to $f(x)$ at almost every point $x \in \mathbb{R}^n$.

2 Other Important Stuff

Theorem 2.1. ***Divergence Theorem***
http://en.wikipedia.org/wiki/Divergence_theorem

$$\int_{\Omega} (\nabla \cdot \mathbf{F})\, dV = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n})\, dS$$

Theorem 2.2. ***Mean Value Theorem***
http://en.wikipedia.org/wiki/Mean_value_theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Definition 2.3. ***Laplacian Operator for a Radial Function***
http://mathworld.wolfram.com/Laplacian.html

For a radial function $g(x)$, the Laplacian is

$$\Delta g = \frac{2}{r}\frac{dg}{dr} + \frac{d^2 g}{dr^2}$$

Theorem 2.4. ***Green's Theorem***
http://en.wikipedia.org/wiki/Green%27s_theorem

Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ (thus, Q is vector-valued). Then

$$\int_{\Omega} \text{div } Q\, dV = \int_{\partial\Omega} Q \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is the outward unit normal. Also,

$$\int_{\Omega} (u\Delta v - v\Delta u) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right)$$

Definition 2.5. ***Divergence***
http://en.wikipedia.org/wiki/Divergence

Let $Q : \mathbb{R}^n \to \mathbb{R}^n$, $Q(\mathbf{x}) = (Q_1(\mathbf{x}), Q_2(\mathbf{x}), \ldots, Q_n(\mathbf{x}))$. Then the divergence operator div : $\mathbb{R}^n \to \mathbb{R}$ is defined by

$$\text{div } Q = \nabla \cdot Q = \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + \cdots + \frac{\partial Q_n}{\partial x_n}$$

Note that the Laplacian operator can be rewritten as

$$\Delta = \text{div } \cdot \text{grad}$$

3 Summaries

3.1 Chapter 1: L^p Spaces

This Chapter begins by defining an L^p space and then introduces key theorems from measure theory (see the "Measure Theory" section). First we look at the L^p spaces, $1 \leq p < \infty$. Using these measure theory results, we prove that the L^p spaces are Banach spaces (i.e. complete normed linear spaces). For a sequence of functions (f_n), we remark that: convergence in $L^p(X)$ $\not\Leftrightarrow$ pointwise convergence a.e. However, it is true that if $f_n \to f$ pointwise a.e. and $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ in $L^p(X)$. Next, we prove that $L^\infty(X)$ is a Banach space.

Now we consider L^p vs. L^q. In general, there is no inclusion relation. For example, if $f(x) = \frac{1}{\sqrt{x}}$, then $f \in L^1(0,1)$ but $f \notin L^2(0,1)$. Conversely, if $f(x) = \frac{1}{x}$, then $f \in L^2(1,\infty)$ but $f \notin L^1(1,\infty)$. We then discuss density in $L^p(X)$. We define mollifiers (see the Mollifiers section), the open subset

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\},$$

and the set

$$L^p_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \in L^p(\tilde{\Omega}) \quad \forall\, \tilde{\Omega} \subset\subset \Omega\}.$$

p	Functions	Are dense in...
$1 \leq p < \infty$	Simple functions, $f = \sum_{i=1}^n a_i \mathbf{1}_{E_i}$	$L^p(X)$
$1 \leq p < \infty$	$C^0(\Omega) = C(\Omega)$	$L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$ bounded
$1 \leq p < \infty$	$C^\infty(\Omega_\epsilon)$ (i.e. f^ϵ)	$L^p_{\text{loc}}(\Omega)$

Next, we define the dual space and present the Riesz representation theorem. Note that $L^1(X) \subset L^\infty(X)'$, and the inclusion is strict. We define what it means for a sequence of linear functionals (ϕ_j) to converge in the weak-$*$ topology.

Definition 3.1. ***Weak Convergence, Weak-$*$ Convergence***
Hunter's 218 Notes (page 7)

A sequence (x_n) in X converges weakly to $x \in X$, written $x_n \rightharpoonup x$, if $(\omega, x_n) \to (\omega, x)$ for every $\omega \in X^*$. A sequence (ω_n) in X^* converges weak-$*$ to $\omega \in X^*$, written $\omega_n \overset{*}{\rightharpoonup} \omega$, if $(\omega_n, x) \to (\omega, x)$ for every $x \in X$.

If X is reflexive, meaning that $X^{**} = X$, then weak and weak-$*$ convergence are equivalent.

Alaoglu's Lemma tells us that for a Banach space $\mathcal{B}$, the closed unit ball in $\mathcal{B}'$ is weak-$*$ compact. For $1 \leq p < \infty$, we define what it means for a sequence of functions (f_n) to converge weakly. Next, we claim that for $1 < p < \infty$, $L^p(X)$ is weak compact: for a bounded subsequence (f_n), there exists a weakly convergent subsequence f_{n_k}. For $p = \infty$, we have that $L^\infty(X)$ is weak-$*$ compact. A simple result using Hölder's inequality is that L^p convergence implies weak convergence. We also prove that if $f_n \rightharpoonup f$ in L^p, then $\{\|f_n\|_p\}$ is bounded (uniform boundedness theorem) and $\|f\|_p \leq \liminf \|f_n\|_p$. We conclude this chapter with Young's inequality:

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \qquad \text{where } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

3.2 Chapter 2: The Sobolev Spaces $H^k(\Omega)$ for Integers $k \geq 0$

We begin by defining the space of test functions, $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, and from this we get the integration by parts formula. We define the Sobolev spaces, $W^{k,p}(\Omega)$, and the special case $H^k(\Omega) = W^{k,2}(\Omega)$. We prove that these are Banach spaces.

Next we want to approximate $W^{k,p}(\Omega)$ functions by smooth functions. We prove that $u^\epsilon \in C^\infty(\Omega_\epsilon)$ for all $\epsilon > 0$, and that $u^\epsilon \to u$ in $W^{k,p}_{\text{loc}}(\Omega)$ as $\epsilon \to 0$.

We introduce the Hölder spaces, which interpolate between $C^0(\overline{\Omega})$ and $C^1(\overline{\Omega})$. For $0 < \gamma \leq 1$, the $C^{0,\gamma}(\overline{\Omega})$ Hölder space consists of the functions

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})} < \infty,$$

$$\text{where} \qquad [u]_{C^{0,\gamma}(\overline{\Omega})} := \max_{\substack{x,y\in\Omega \\ x\neq y}} \left(\frac{|u(x) - u(y)|}{|x-y|^\gamma} \right).$$

We have that $C^{0,\gamma}(\overline{\Omega})$ is a Banach space.

We prove that if a function has a weak derivative, then it is differentiable a.e. and its weak derivative equals its classical derivative a.e. We define the space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. We define $H^{-1}(\Omega)$ as the dual space of $H_0^1(\Omega)$.

Theorems covered include:

- Sobolev Embedding Theorem (2-D)
- Morrey's Inequality
- Sobolev Embedding Theorem ($k = 1$)
- Gagliardo-Nirenberg Inequality
- Poincaré Inequalities
  - Gagliardo-Nirenberg Inequality for $W^{1,p}(\Omega)$
  - Gagliardo-Nirenberg Inequality for $W_0^{1,p}(\Omega)$
- Rellich's Theorem

3.3 Chapter 3: The Fourier Transform

We begin by defining the Fourier transform, $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$$

and its adjoint (equivalently, its inverse for $f \in \mathcal{S}(\mathbb{R}^n)$),

$$\mathcal{F}^* f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi}\, d\xi.$$

Plancherel's Theorem tells us that for $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \mathcal{F}u, \mathcal{F}v \rangle_{L^2(\mathbb{R}^n)} = \langle u, v \rangle_{L^2(\mathbb{R}^n)}.$$

Here we have used the definition of the space of Schwartz functions (of rapid decay):

$$\begin{aligned}\mathcal{S}(\mathbb{R}^n) &= \{u \in C^\infty(\mathbb{R}^n) \mid x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n)\ \forall\ \alpha, \beta \in \mathbb{Z}_+^n\} \\ &= \{u \in C^\infty(\mathbb{R}^n) \mid \langle x \rangle^k |D^\alpha u| \leq C_{k,\alpha}\ \forall\ k \in \mathbb{Z}_+\}, \qquad \text{where } \langle x \rangle = \sqrt{1+|x|^2}.\end{aligned}$$

We note that $\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. The second equality motivates the definition of the semi-norm

$$p_k(u) = \sup_{x \in \mathbb{R}^n,\ |\alpha| \leq k} \langle x \rangle^k |D^\alpha u(x)|$$

and the metric

$$d(u, v) = \sum_{k=0}^\infty 2^{-k} \frac{p_k(u-v)}{1 + p_k(u-v)}$$

on $\mathcal{S}(\mathbb{R}^n)$. We say that a sequence $u_j \to u$ in $\mathcal{S}(\mathbb{R}^n)$ if $p_k(u_j - u) \to 0$ for all $k \in \mathbb{Z}_+$. We define the space of tempered distributions as $\mathcal{S}'(\mathbb{R}^n)$, i.e., the set of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. We define the distributional derivative $D : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by

$$\begin{aligned}\langle DT, u \rangle &= -\langle T, Du \rangle \quad \forall\ u \in \mathcal{S}(\mathbb{R}^n) \\ \langle D^\alpha T, u \rangle &= (-1)^{|\alpha|} \langle T, D^\alpha u \rangle \quad \forall\ u \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$

We define the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$, $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, by

$$\langle \mathcal{F}T, u \rangle = \langle T, \mathcal{F}u \rangle \quad \forall\ u \in \mathcal{S}(\mathbb{R}^n),$$

and similarly for $\mathcal{F}^*$. Using the density of $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, we extend the Fourier transform to $L^2(\mathbb{R}^n)$. We prove the Hausdorff-Young Inequality and the Riemann-Lebesgue Lemma. We prove two theorems regarding the Fourier transforms of convolutions. First, if $u, v \in L^1(\mathbb{R}^n)$ then $u * v \in L^1(\mathbb{R}^n)$ and

$$\mathcal{F}(u * v) = (2\pi)^{n/2} \mathcal{F}u \mathcal{F}v.$$

The second result generalizes the first: suppose $1 \leq p, q, r \leq 2$ satisfy $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. Then for $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, $\mathcal{F}(u * v) \in L^{\frac{r}{r-1}}(\mathbb{R}^n)$, and

$$\mathcal{F}(u * v) = (2\pi)^{n/2} \mathcal{F}u \mathcal{F}v.$$

3.4 Chapter 4: The Sobolev Spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$

We begin by defining the Sobolev spaces $H^s(\mathbb{R}^n)$, where s is not restricted to the integers, as

$$\begin{aligned}H^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^n)\} \\ &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \Lambda^s u \in L^2(\mathbb{R}^n)\},\end{aligned}$$

where $\langle \xi \rangle = \sqrt{1+|\xi|^2}$ and $\Lambda^s u = \mathcal{F}^*(\langle \xi \rangle^s \hat{u})$. We define an inner product on $H^2(\mathbb{R}^n)$ as

$$\langle u, v \rangle_{H^s(\mathbb{R}^n)} = \langle \Lambda^s u, \Lambda^s v \rangle_{L^2(\mathbb{R}^n)} \qquad \forall\ u, v \in H^2(\mathbb{R}^n),$$

and the norm is defined accordingly. We have that for all $s \in \mathbb{R}$, $[H^s(\mathbb{R}^n)]' = H^{-s}(\mathbb{R}^n)$.

3.5 Chapter 5: Fractional-Order Sobolev spaces on Domains with Boundary

3.6 Chapter 6: The Sobolev Spaces $H^s(\mathbb{T}^n),\ s \in \mathbb{R}$

For $u \in L^1(\mathbb{T}^n)$ and $k \in \mathbb{Z}^n$, we define

$$\mathcal{F}u(k) = \hat{u}_k = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-ik\cdot x} u(x)\, dx$$
$$\mathcal{F}^* u(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}_k e^{ik\cdot x}$$

We let $\mathfrak{s} = \mathcal{S}(\mathbb{Z}^n)$ denote the space of rapidly decreasing functions $\hat{u}$ on $\mathbb{Z}^n$, where

$$p_N(u) = \sup_{k \in \mathbb{Z}^n} \langle k \rangle^N |\hat{u}_k| < \infty \qquad \forall N \in \mathbb{N}.$$

4 Things That Are Inescapable

- Dominated Convergence Theorem (DCT)
- Monotone Convergence Theorem (MCT)
- Convolutions
- Green's Theorem

5 Tricks & Techniques

- when $\Omega = B(0,1)$, define $B_\delta = B(0,1) - B(0,\delta)$
- FTC to get a difference
- FTC to get $u(x)$ from $\partial_j u(x)$
- polar coordinates
- (Assume that) the weak derivative is equal to the classical derivative almost everywhere
- Use that if
$$\int_\Omega u(x)\phi(x)\,dx = 0 \quad \forall\ \phi \in C_0^\infty(\Omega)$$
then $u = 0$ a.e. in Ω.
- Choose your coordinate system centered around x, which allows us to assume $x = 0$
- Use an indicator function to allow us to extend the integral to a bigger region
- Identify potential singularities and rule them out (e.g. by L'Hospital's rule)
- Cut-off functions, such as
$$g(x) = \begin{cases} 1 & x \in \left[0, \frac{1}{2}\right] \\ 0 & x \in \left[\frac{3}{4}, \infty\right) \end{cases}$$
- $\partial_{x_j}\eta_\epsilon(x-y) = -\partial_{y_j}\eta_\epsilon(x-y)$
- Integrate from $-\infty$ to x or from 0 to x

6 Mollifiers

Standard Mollifier

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$
$$\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right)$$

Indicator Mollifier

$$\frac{1}{h}\mathbf{1}_{[0,h]}$$

Poisson Kernel

$$p_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

From HW3

$$\eta(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2}$$
$$\eta_\epsilon(x) = \frac{1}{\pi}\cdot\frac{\epsilon}{\epsilon^2+\xi^2}$$

7 Inequalities

Theorem 7.1. ***Sobolev ($n = 2$)***
page 30

For $kp \geq 2$,

$$\max_{x \in \mathbb{R}^2} |u(x)| \leq C\|u\|_{W^{k,p}(\mathbb{R}^2)}$$

Theorem 7.2. ***Sobolev ($k = 1$)***
page 36

Implied by Morrey's Inequality.

$$\|u\|_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Theorem 7.3. ***Morrey's Inequality***
page 33

"A refinement and extension of Inequality 7.1 (Sobolev for $n = 2$)."

For $n < p \leq \infty$:

$$|u(x) - u(y)| \leq Cr^{1-n/p}\|Du\|_{L^p(B(x,2r))} \qquad \forall\ u \in C^1(\mathbb{R}^n)$$

Contrast with: Gagliardo-Nirenberg Inequality 7.4.

Theorem 7.4. ***Gagliardo-Nirenberg***
page 38

For $1 \leq p < n$:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{p,n}\|Du\|_{L^p(\mathbb{R}^n)}$$

where

$$p^* = \frac{np}{n-p}.$$

This holds for every $u \in W^{1,p}(\mathbb{R}^n)$ $\Leftarrow$ since we need at least 1 derivative.

Contrast with: Morrey's Inequality 7.3.

Theorem 7.5.
page 41

For $1 \leq q < \infty$:

$$\|u\|_{L^1(\mathbb{R}^2)} \leq C\sqrt{q}\|u\|_{H^1(\mathbb{R}^2)}$$

where $u \in H^1(\mathbb{R}^2)$.

Compare to: Theorem 7.8.

Theorem 7.6. ***Gagliardo-Nirenberg for*** $W^{1,p}(\Omega)$
page 46

For $1 \leq p < n$:

$$\|u\|_{L^{p^*}(\Omega)} \leq C_{p,n,\Omega}\|u\|_{W^{1,p}(\Omega)}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary.

Theorem 7.7. ***Poincaré 1*** $\equiv$ ***Gagliardo-Nirenberg for*** $W_0^{1,p}(\Omega)$
page 46

For $1 \leq p < n$ and $1 \leq q \leq p^*$:

$$\|u\|_{L^q(\Omega)} \leq C_{p,n,\Omega}\|Du\|_{L^p(\Omega)}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary.

Theorem 7.8. ***Poincaré 2***
page 46

For all $1 \leq q < \infty$:

$$\|u\|_{L^q(\Omega)} \leq C_\Omega\sqrt{q}\|Du\|_{L^2(\Omega)}$$

where $\Omega \subset \mathbb{R}^2$ is open and bounded with a C^1 boundary.

Compare to: Theorem 7.5.

Remark 7.9. ***Inequality Overview***

- Sobolev Inequalities: 7.1 and 7.2
- Morrey's Inequality: 7.3
- Gagliardo-Nirenberg Inequality (Main): 7.4
  - Gagliardo-Nirenberg Inequalities (Secondary): 7.6 and 7.7
- Poincaré Inequalities: 7.7 and 7.8

8 Definitions

Definition 8.1. ***Weak & Weak-* Convergence***
page 18

If

$$\int_X f_n \phi(x)\,dx \to \int_X f(x)\phi(x)\,dx \qquad \forall\ \phi \in L^q(X), \quad q = \frac{p}{p-1}$$

then

- $(p \neq \infty)$ $f_n \rightharpoonup f$ in $L^p(X)$ weakly.
- $(p = \infty)$ $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(X)$ weak-*.

The reason for this distinction is because $L^\infty(\Omega)' \neq L^1(\Omega)$. Rather, $L^\infty(\Omega)' = \mathcal{M}(\Omega) =$ Radon Measures.

Theorem 8.2. ***Weak Compactness of L^p / Weak-* Compactness of L^∞***
page 18

Given a bounded sequence $(f_n) \subset L^p(X)$, there exists a

- weakly convergent subsequence if $1 < p < \infty$.
- weak-* convergent subsequence if $p = \infty$.

I suspect that the reason why L^1 is not weakly compact has to do with the fact that $L^\infty(\Omega)' \subset L^1(\Omega)$, where the inclusion is strict.

Definition 8.3. ***Sobolev Norm***
page 29

For $p \neq \infty$:

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}$$

For $p = \infty$:

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

Definition 8.4. ***Embed***
page 30

For 2 Banach spaces, B_1 and B_2, we say that B_1 is embedded in B_2, denoted $B_1 \hookrightarrow B_2$, if

$$\|u\|_{B_2} \leq C\|u\|_{B_1} \quad \forall\ u \in B_1.$$

The intuition is that for norms of a similar structure, every $u \in B_1$ will automatically be in B_2.

Definition 8.5. ***Standard Mollifier***
page 32

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$
$$\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right)$$

Definition 8.6. ***Hölder Norm***
page 33

$$\|u\|_{C^0(\overline{\Omega})} = \max_{x\in\Omega} |u(x)|$$
$$\|u\|_{C^1(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + \|Du\|_{C^0(\overline{\Omega})}$$

Definition 8.7. ***Hölder Semi-Norm***
page 33

For $0 < \gamma \leq 1$, we define
$$[u]_{C^{0,\gamma}(\overline{\Omega})} = \max_{\substack{x,y\in\Omega \\ x\neq y}} \left(\frac{|u(x)-u(y)|}{|x-y|^\gamma}\right).$$
We also define
$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}.$$

Definition 8.8. $W_0^{1,p}(\Omega)$

$$W_0^{1,p}(\Omega) \triangleq \text{the closure of } C_0^\infty(\Omega) \text{ in } W^{1,p}(\Omega)$$

Definition 8.9. $H^{-1}(\Omega)$

$$H^{-1}(\Omega) \triangleq \text{the dual space of } H_0^1(\Omega)$$

Definition 8.10. ***Fourier Transform***
page 55

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$$

Definition 8.11. ***Inverse Fourier Transform***
page 56

$$\mathcal{F}^* f(x) = \check{\hat{f}}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi}\, d\xi$$

Theorem 8.12. ***Plancherel's Theorem***
page 58

$$(\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{F}^*\mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)}$$

Definition 8.13. ***Gaussian***
page 58

$$G(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$$
$$\hat{G}(\xi) = (2\pi)^{-n/2} e^{-\xi^2/2}$$

Definition 8.14. ***Schwartz Functions of Rapid Decay***
page 55

$$\begin{aligned}\mathcal{S}(\mathbb{R}^n) &= \left\{ u \in C^\infty(\mathbb{R}^n) \;\middle|\; x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n)\ \forall\ \alpha, \beta \in \mathbb{Z}_+^n \right\} \\ &= \left\{ u \in C^\infty(\mathbb{R}^n) \;\middle|\; \langle x \rangle^k |D^\alpha u| \le C_{k,\alpha} \quad \forall\ k \in \mathbb{Z}_+ \right\}\end{aligned}$$

where

$$\langle x \rangle = \sqrt{1 + |x|^2}.$$

The prototypical element of $\mathcal{S}(\mathbb{R}^n)$ is $e^{-|x|^2}$.

Definition 8.15. ***$\mathcal{S}(\mathbb{R}^n)$ Semi-Norm and Metric***
page 59

For $k \in \mathbb{Z}_+$ we have the semi-norm:

$$p_k(u) = \sup_{x\in\mathbb{R}^n,\ |\alpha|\le k} \langle x\rangle^k |D^\alpha u(x)|.$$

We have the metric:

$$d(u,v) = \sum_{k=0}^{\infty} 2^{-k} \frac{p_k(u-v)}{1+p_k(u-v)}.$$

Definition 8.16. ***Distributional Derivative on $\mathcal{S}'(\mathbb{R}^n)$***
page 60

$$\langle D^\alpha T, u\rangle = (-1)^{|\alpha|} \langle T, D^\alpha u\rangle \qquad \forall\ u \in \mathcal{S}(\mathbb{R}^n)$$

Examples:

$$\left\langle \frac{dH}{dx}, u \right\rangle = \langle \delta, u\rangle$$
$$\left\langle \frac{d\delta}{dx}, u \right\rangle = -\frac{du}{dx}(0)$$

Definition 8.17. ***Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$***
page 60

$$\langle \mathcal{F}T, u\rangle = \langle T, \mathcal{F}u\rangle \qquad \forall\ u \in \mathcal{S}(\mathbb{R}^n)$$

Examples:

$$\mathcal{F}\delta = (2\pi)^{-n/2}$$
$$\mathcal{F}^*\delta = (2\pi)^{-n/2}$$
$$\mathcal{F}^*\left[(2\pi)^{n/2}\right] = 1$$

Theorem 8.18. ***Fourier Transform of a Convolution***
page 63

$$\mathcal{F}(u*v) = (2\pi)^{n/2}\mathcal{F}u\mathcal{F}v$$
$$\widehat{u*v} = (2\pi)^{n/2}\hat{u}\hat{v}$$

Definition 8.19. ***General Hilbert Space:*** $H^s(\mathbb{R}^n)$
page 74

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \langle \xi \rangle^s \, \hat{u} \in L^2(\mathbb{R}^n) \right\}$$

Thus, $H^{1/2}(\mathbb{R}^n)$ is the space of L^2 functions with 1/2 a derivative, and $H^{-1}(\mathbb{R}^n)$ is the space of functions whose anti-derivative is in L^2.

Definition 8.20. ***Poisson Integral Formula***
page 89

The Poisson Integral Formula is

$$\mathrm{PI}(f)(r,\theta) = \sum_{k \in \mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta}$$

and it satisfies

$$\begin{aligned} \Delta \mathrm{PI}(f) &= 0 \text{ in } D \\ \mathrm{PI}(f) &= f \text{ on } \partial D = \mathbb{S}^1 \end{aligned}$$

A 3-28-11

Remark A.1.

In general, Ω will be used to represent a smooth, open subset. That is, $\Omega \subset \mathbb{R}^d$, open.

Lemma A.2.

Let $\Omega \subset \mathbb{R}^d$ be open. Suppose $u \in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega u(x)v(x)\,dx = 0 \quad \forall\, v \in C_0^\infty(\Omega)$$

(Recall: $C_0^\infty(\Omega)$ is the set of functions that are infinitely differentiable and have compact support in Ω.) Then $u = 0$ a.e. in Ω.

Proof. If $\int_\Omega |u|\,dx = 0$ then $u = 0$ a.e. in Ω. Consider the sign function, and note that $|u| = \text{sgn}(u)$. We want to approximage sgn with C^∞ functions. Choose $g \in L^\infty(\mathbb{R}^d)$ with supp $g = \text{spt } g \subset \Omega$, and for the sake of simplicity suppose that the support of g is compact. (Note: in this case, we are going to set $g(x) = \text{sgn}(x)$.) Approximate g via convolution with an approximate identity. Let ρ_ϵ be a smooth approximate identity with $\int \rho_\epsilon\,dx = 1$ and with support in $B(0, \epsilon)$. Define

$$g^\epsilon = \rho_\epsilon * g$$

Then

$$g^\epsilon(x) = \int_{\mathbb{R}^d} \rho_\epsilon(x-y)g(y)\,dy = \int_{B(x,\epsilon)} \rho_\epsilon(x-y)g(y)\,dy \quad \text{(by DCT)}$$

Convolution theory gives us that

1. $g^\epsilon \in C_0^\infty(\Omega)$. C^∞ is given by the DCT, and we achieve compact support in Ω by taking ϵ sufficiently small.
2. $g^\epsilon \to g$ in $L^2(\Omega)$ as $\epsilon \searrow 0$ implies that $g^{\epsilon'} \to g$ a.e. (See Lemma A.3.)

Lemma A.3.

If $g^\epsilon \to g$ in $L^2(\Omega)$, then there exists a subsequence $g^{\epsilon'}(x) \to g(x)$ a.e. in Ω.

Definition A.4. *L^1 Convergence*

$u_j \to u$ in $L^1(\Omega)$ if $\|u_j - u\|_{L^1(\Omega)} \to 0 \;\Leftrightarrow\; \int_\Omega |u_j - u|\,dx \to 0$.

From above, (1) implies that $\int_\Omega u(x)g^\epsilon(x)\,dx = 0$. (2) implies that $\int_\Omega u(x)g(x)\,dx = 0$ by the DCT. To complete the proof, let $K^{\text{cpt}} \subset \Omega$ and choose $g = \text{sgn}(u)$ with support on K. Then $\int_K |u|\,dx = 0$, and so $u = 0$ a.e. in K. K is arbitrary, so $u = 0$ a.e. in Ω. ∎

3 (or 2?) Steps To Proving Lemma A.3 (For proof see Example B.1)

1. Restrict to a subsequence g_k such that

$$\|g_{k+1} - g_k\|_{L^p(\Omega)} \leq \frac{1}{2^k}$$

   Using this bound, the goal is to convert from Cauchy in L^p to Cauchy pointwise a.e.

2. Conversion to a monotone sequence:

$$q_1 = 0, \quad q_2 = |g_2 - g_1| + |g_1|, \quad q_3 = |g_3 - g_2| + |g_2 - g_1| + |g_1|$$

$$q_n = \sum_{l=1}^{n-1} |g_{l+1} - g_l| + |g_1|$$

   Then $0 \leq q_1 \leq q_2 \leq q_3 \leq \ldots$, so we have a monotonically increasing sequence, $q_n \in L^p$, and by the MCT we get that $q_n \nearrow q \in L^p$

B Section 3-29-11

Example B.1.

Given: $(g_n) \subset L^1(X),\ g_n \to g$ in $L^1(X) \ \Rightarrow \ \lim_{n\to\infty} \|g_n - g\|_{L^1} = 0$
Prove: There exists a subsequence (g_{n_j}) such that $g_{n_j} \to g$ pointwise a.e.

Proof. Construct a pointwise Cauchy subsequence.

Aside: Consider a sequence (a_n) that satisfies $a_n \leq a_{n+1} \leq \ldots$.
If it is bounded then it is convergent, and hence Cauchy.
If it is unbounded then it is not convergent.

Since $\lim_{n\to\infty} \|g_n - g\|_{L^1} = 0$, the sequence is convergent, so it is bounded, so there exists M such that $\|g_n\|_{L^1} \leq M$. We can choose a subsequence (g_{n_j}) such that

$$\|g_{n_j} - g_{n_{j-1}}\| \leq \frac{1}{2^j}$$

Now we construct a function $h_j(x)$ that is a sum of measurable functions:

$$h_j(x) = |g_{n_1}(x)| + \sum_{k=2}^{j} \left|g_{n_k}(x) - g_{n_{k-1}}(x)\right|$$

We can bound the L^1 norm of each h_j:

$$\|h_j\|_{L^1} \leq \|g_{n_1}\|_{L^1} + C$$

By the Monotone Convergence Theorem, $\lim_{j\to\infty} h_j(x) = h(x)$ (pointwise limit a.e.) $\in L^1(X)$ and $\|h_j - h\| \to 0$.

The sequence $\big(h_j(x)\big)$ is Cauchy a.e. Therefore, $\big(g_{n_j}(x)\big)$ is Cauchy a.e. because

$$\left|g_{n_j}(x) - g_{n_k}(x)\right| \leq h_j(x) - h_k(x), \quad j \geq k$$

Therefore, $\lim_{j\to\infty} g_{n_j}(x) = g'(x)$. We know that

$$\begin{aligned} \left|g_{n_j}(x)\right| &\leq h_j(x) \quad \forall\ j \\ |g'(x)| &\leq h(x) \end{aligned} \tag{B.1}$$

However, we don't know that the pointwise limit g' is the same as the strong limit g. We must show that g' is the strong limit of (g_{n_j}). Expanding on (B.1), we write

$$\left|g_{n_j}(x)\right| \leq h_j(x) \leq h(x) \quad \forall\ j$$

Use the Lebesgue Dominated Convergence Theorem to show that $g' = g$ a.e.:

$$\lim_{n\to\infty} \int \left|g_{n_j} - g'\right| \, dx = 0 = \lim_{n\to\infty} \left\|g_{n_j} - g'\right\| = 0, \qquad \left|g_{n_j} - g'\right| \leq 2h$$

□

Remark B.2. ***3 Important Theorems from Measure Theory***

- Monotone Convergence Theorem
- Lebesge Dominated Convergence Theorem
- Fatou's Lemma

Example B.3. ***MCT ⇒ Fatou's Lemma***

Recall: Fatou's Lemma states that:

$$\int_\Omega \liminf_{n\to\infty} f_n \, dx \le \liminf_{n\to\infty} \int_\Omega f_n(x) \, dx$$

Proof. Start with the definition of lim inf. For a given sequence (a_n), let

$$x_n = \inf_{m\ge n} a_m$$

(x_n) is an increasing sequence, and

$$\lim_{n\to\infty} x_n = \begin{cases} \text{exists} = \liminf_{n\to\infty} a_n \\ \infty \end{cases}$$

Assume that $f_n(x) \ge 0 \ \forall \ n$. Define

$$g_n(x) = \inf_{m\ge n} f_m(x) \ge 0 \tag{B.2}$$

g is measurable, and

$$0 \le g_1(x) \le g_2(x) \le g_3(x) \le \dots$$

Somehow we get

$$\lim_{n\to\infty} \int_\Omega g_n(x) \, dx = \int_\Omega \liminf_{n\to\infty} f_n(x) \, dx$$
$$\int_\Omega g_n(x) \, dx \le \inf_{m\ge n} \int_\Omega f_m(x) \, dx$$
$$\int_\Omega \liminf_{n\to\infty} f_n(x) \, dx \le \liminf_{n\to\infty} \int_\Omega f_n(x) \, dx$$

□

Example B.4. ***Fatou's Lemma ⇒ LDCT***

Given: $f_n(x) \to f(x)$ a.e., $|f_n(x)| \le g(x)$, where $g \in L^1(X)$
Prove: $f \in L^1(X)$ and $\lim_{n\to\infty} \int_X f_n(x) \, dx = \int_X f(x) \, dx$

Proof. First, show that $f \in L^1$. Integrating the inequality $|f_n(x)| \leq g(x)$ gives us

$$\int_X |f_n(x)|\, dx \leq \int_X g(x)\, dx$$

Taking the limit as $n \to \infty$, we get that

$$\int_X |f(x)|\, dx \leq \underbrace{\liminf_{n\to\infty}}_{\limsup?} \int_X |f_n(x)|\, dx \leq \int_X g(x)\, dx$$

So $f \in L^1$.

Define

$$h_n = g \pm f_n \geq 0$$

Adding:

$$\begin{aligned} \int g + f\, dx &\leq \liminf_{n\to\infty} \left(\int g\, dx + \int f_n\, dx \right) \\ &\leq \int g\, dx + \liminf_{n\to\infty} \int f_n\, dx \\ \int f\, dx &\leq \liminf_{n\to\infty} \int f_n\, dx \end{aligned}$$

where the simplification from the first line to the second is allowed because g is constant, so $\int (g+f)\ dx = \int f\, dx$.

Subtracting:

$$\begin{aligned} \int g - f\, dx &\leq \liminf_{n\to\infty} \left(\int g\, dx - \int f_n\, dx \right) \\ -\int f\, dx &\leq \liminf_{n\to\infty} \left(- \int f_n\, dx \right) \\ \int f\, dx &\geq \limsup_{n\to\infty} \int f_n\, dx \end{aligned}$$

where the change form the second line to the third is because $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$. Thus, we have

$$\limsup_{n\to\infty} \int f_n\, dx \leq \int f\, dx \leq \liminf_{n\to\infty} \int f_n\, dx \leq \limsup_{n\to\infty} \int f_n\, dx$$

and therefore

$$\lim_{n\to\infty} \int_X f_n(x)\, dx = \int_X f(x)\, dx$$

□

C 3-30-11

Definition C.1. ***L^p Spaces***

Given $\Omega \subset \mathbb{R}^d$ open and smooth, we define

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} \bigm| \|u\|_{L^p(\Omega)} < \infty\}$$
$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \bigm| |u(x)| \leq C \text{ a.e. } \}$$
$$\|u\|^p_{L^p(\Omega)} = \int_\Omega |u(x)|^p \, dx \quad 1 \leq p < \infty$$

Remark C.2.

Fact: for $1 \leq p \leq \infty$, $L^p(\Omega)$ is a vector space.

Definition C.3. ***Conjugate Exponent***

For $1 \leq p \leq \infty$, we define the *conjugate exponent* q such that

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad q = \frac{p}{p-1}$$

Theorem C.4. ***Hölder's Inequality***

If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

Theorem C.5. ***Minkowski's Inequality***

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Corollary C.6.

$L^p(\Omega)$ is a normed vector space.

Fact: L^p is a Banach space.

Theorem C.7.

For $1 \le p < \infty$, $C_0(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

Lemma C.8.

On a bounded domain, i.e. $|\Omega| < \infty$, for $1 \le p \le q \le \infty$, we have $L^q \subset L^p$ with continuous injection, and

$$\|u\|_{L^p(\Omega)} \le |\Omega|^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^q(\Omega)}$$

Proof. (Sample)

$$\int_\Omega u(x)\,dx = \int_\Omega u(x)\cdot 1\,dx \le \left(\int_\Omega 1\,dx\right)^{1/2} \left(\int_\Omega |u(x)|^2\,dx\right)^{1/2}$$

(By Hölder's Inequality) □

Problem C.9.

Prove $L^1(\Omega) \cap L^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p \le \infty$.

Proof.

$$\Omega = \cup_{n=1}^\infty \Omega_n \text{ with } |\Omega_n| < \infty$$
$$u \in L^p(\Omega) \Rightarrow u_n = \mathbf{1}_{\Omega_n} t_n(u)$$

□

Definition C.10. ***Indicator Function***

$$\mathbf{1}_E = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

Definition C.11. ***Truncation Operator***

$$t_M(u) = \begin{cases} u & \text{if } |u| \le M \\ M\frac{u}{|u|} & \text{if } |u| > M \end{cases}$$

Problem C.12.

Prove $u \in L^2(\Omega) \cap L^1(\Omega) \mid \|u\|_{L^1(\Omega)} \le 1\}$ is closed in $L^2(\Omega)$.

Proof.

$$u_n \to u, \; u_n \in L^1 \cap L^2$$
$$u_{n_k} \to u(x) \text{ a.e. in } \Omega$$
$$\int_\Omega |u(x)|\, dx \le \liminf \int_\Omega |u_{n_k}|\, dx \le 1 \quad \text{(Fatou's Lemma)}$$

□

Definition C.13. ***Compactly Contained*** $(\subset\subset)$

$\Omega_1 \subset\subset \Omega \Leftrightarrow \Omega_1 \subset K^{\text{cpt}} \subset \Omega$
We say that Ω_1 is *compactly contained* in Ω.

Definition C.14. $L^p_{loc}(\Omega)$

$L^p_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \in L^p(\tilde{\Omega}) \; \forall\, \tilde{\Omega} \subset\subset \Omega\}$

Definition C.15. Ω_ϵ

$\Omega_\epsilon = \{x \in \Omega \mid d(x, \partial\Omega) > \epsilon\}$

Definition C.16. ***Mollifier***
`http://en.wikipedia.org/wiki/Mollifier`

Mollifiers are smooth functions with special properties, used in distribution theory to create sequences of smooth functions approximating nonsmooth (generalized) functions, via convolution. For example,

$$\rho(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \ge 1 \end{cases} \qquad \rho(x) \ge 0$$

$$\int_{\mathbb{R}^d} \rho(x)\, dx = 1, \qquad \rho \in C_0^\infty(\mathbb{R}^d), \qquad \text{spt } (\rho) \subset B(0,1)$$

Definition C.17. ***Dilated Family***

$$\rho_\epsilon(x) = \frac{1}{\epsilon^d}\rho\left(\frac{x}{\epsilon}\right)$$

It follows that

$$\int_{\mathbb{R}^d} \rho_\epsilon(x)\,dx = 1, \qquad \text{spt }(\rho_\epsilon) \subset B(0,\epsilon)$$

Definition C.18. f^ϵ

For $f \in L^1_{\text{loc}}(\Omega)$, set $f^\epsilon = \rho_\epsilon * f$.

Note: $f^\epsilon : \Omega^\epsilon \to \mathbb{R},\ \epsilon > 0$.

Theorem C.19.

For $f^\epsilon \in C^\infty(\Omega^\epsilon)$, $f^\epsilon(x) \to f(x)$ a.e. $f \in C(\overline{\Omega}) \Rightarrow f^\epsilon \to f$ uniformly on compact (?). If $f \in L^p(\Omega)$, $p \in [0,\infty)$ then $f^\epsilon \to f$ in $L^p(\Omega)$.

Proof. Choose h small such that $x + he_i \in \Omega$, where e_i is a basis vector of $\mathbb{R}^d$. Consider

$$\frac{f^\epsilon(x+he_i) - f^\epsilon(x)}{h} = \underbrace{\frac{\int_{\mathbb{R}^d} \rho_\epsilon(x+he_i-y) - \rho_\epsilon(x-y)f(y)\,dy}{h}}$$

The underbraced term is bounded by $\frac{1}{\epsilon}\frac{\partial\rho_\epsilon}{\partial x_i}$ by the Mean Value Theorem. So by the DCT, we can pass to the limit as $h \searrow 0$. $\square$

D 4-1-11

Theorem D.1.

$f^\epsilon \to f$ in $L^p_{\text{loc}}(\Omega)$

Proof.

$$\begin{aligned} |f^\epsilon(x) - f(x)| &= \int_{B(x,\epsilon)} \rho_\epsilon(x-y)|f(x)-f(y)|\,dy \\ &= \frac{1}{\epsilon^d}\rho\left(\frac{x-y}{\epsilon}\right)|f(x)-f(y)|\,dy \end{aligned}$$

In general, it is true that

$$\frac{c}{\epsilon^d}\int_{B(x,\epsilon)} \rho\left(\frac{x-y}{\epsilon}\right)|f(x)-f(y)|\,dy \le \frac{c}{|B_\epsilon|}\int_{B(x,\epsilon)} |f(x)-f(y)|\,dy \to 0$$

by the Lebesgue Differentiation Theorem (Theorem 1.11). Thus, $f^\epsilon(x) \to f(x)$ a.e.
If f is continuous on Ω then $f^\epsilon \to f$ uniformly on $\tilde{\Omega} \subset\subset \Omega$. The proof relies on showing that $f^\epsilon \in L^p$.

Given: $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$
Want: $\|f^\epsilon\|_{L^p(\Omega_2)} \le C\|f\|_{L^p(\Omega_1)}$

$$\begin{aligned} |f^\epsilon(x)| &\le \int_{B(x,\epsilon)} \rho_\epsilon(x-y)|f(y)|\,dy \\ &\le \int_{B(x,\epsilon)} \rho_\epsilon(x-y)^{1/q}\rho_\epsilon(x-y)^{1/p}|f(y)|\,dy \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \\ &\le \left(\int_B \rho_\epsilon(x-y)\,dy\right)^{1/q}\left(\int_{B(x,\epsilon)} \rho_\epsilon(x-y)|f(y)|^p\,dy\right)^{1/p} \\ |f^\epsilon(x)|^p &\le \int_{B(x,\epsilon)} \rho_\epsilon(x-y)|f(y)|^p\,dy \\ \int_{\Omega_2} |f^\epsilon(x)|^p\,dx &\le \int_{\Omega_2(x)}\int_{B(x,\epsilon)} \rho_\epsilon(x-y)|f(y)|^p\,dy\,dx \\ &\le \int_{B(x,\epsilon)(y)} |f(y)|^p \int_{\Omega_2(x)} \rho_\epsilon(x-y)\,dx\,dy \\ &\le \int_{\Omega_2(x)}\int_{B(0,\epsilon)(y)} \rho_\epsilon(y)|f(x-y)|^p\,dy\,dx \quad \text{(change of variables)} \\ &\le \int_{B(0,\epsilon)(y)}\int_{\Omega_2(x)} |f(x-y)|^p\,dx\,dy \end{aligned} \tag{D.1}$$

Note that

$$\int_{\Omega_1(x)} |f(x)|^p\,dx = \int_{\Omega_1(y)} |f(y)|^p\,dy = \int_{\Omega_1(y)} |f(y)|^p \underbrace{\int_{B(y,\epsilon)} \rho_\epsilon(x-y)\,dx}_{=1}\,dy$$

We can control (D.1) by integrating over Ω_1.

$C(\Omega_2)$ is dense in $L^p(\Omega_1)$.
Choose $g \in C(\Omega_1)$ such that $\|g - f\|_{L^p(\Omega_1)} \leq \epsilon$. Then

$$\|f - f^\epsilon\|_{L^p(\Omega_2)} \leq \|f - g\|_{L^p(\Omega_2)} + \|g - g^\epsilon\|_{L^p(\Omega_2)} + \|g^\epsilon - f^\epsilon\|_{L^p(\Omega_2)}$$

and

$$\|g^\epsilon - f^\epsilon\|_{L^p(\Omega_2)} = \|\rho_\epsilon * (g - f)\|_{L^p(\Omega_2)} = \|(f - g)^\epsilon\|_{L^p(\Omega_2)}$$

□

Problem D.2.

Let $\rho_{1/n}$ be mollifiers with spt $\rho_{1/n} \subset \overline{B(0, 1/n)}$. Let $u \in L^\infty(\mathbb{R}^d)$ and $z_n \in L^\infty(\mathbb{R}^d)$ such that $z_n(x) \to z(x)$ a.e. and $\|z_n\|_{L^\infty} \leq 1$.
Let $v_n = \rho_{1/n} * z_n u$ and $v = zu$.
Show that $v_n \to v$ in $L^1(B)$ for any ball $B \subset \mathbb{R}^d$, i.e. $\int_B |v_n - v|\,dx \to 0$. Also show $v_n \rightharpoonup v$ in L^∞ weak-$*$.

Proof. Let $B_1 = B(0,1)$, $B_2 = B(0,2)$, $w_n = \rho_{1/n} * \mathbf{1}_{B_2} z_n u$.
Then $v_n = w_n$ on B_1.

$$\int_{B_1} |v_n - v|\,dx = \int_{B_1} |w_n - \mathbf{1}_{B_2} v|\,dx \leq \int_{\mathbb{R}^d} |w_n - \mathbf{1}_{B_2} v|\,dx$$

Finish this using the triangle inequality. □

E 4-4-11

Theorem E.1. ***Riesz Representation Theorem***

Case 1: $1 < p < \infty$
If $\phi \in L^p(\Omega)'$, there exists $u \in L^q(\Omega)$ (where $q = \frac{p}{p-1}$) such that

$$\phi(f) = \int_\Omega uf\,dx \;\forall\; f \in L^p(\Omega), \qquad \|\phi\|_{L^p(\Omega)'} = \|u\|_{L^q(\Omega)}$$

Case 2: $p = 1$
$L^1(\Omega)' = L^\infty(\Omega)$, and the Riesz Representation Theorem states that for every $\phi \in L^1(\Omega)'$ there exists $u \in L^\infty(\Omega)$ such that

$$\phi(f) = \int_\Omega uf\,dx \;\forall\; f, \qquad \|\phi\|_{L^1(\Omega)'} = \|u\|_{L^\infty(\Omega)}$$

Case 3: $p = \infty$
$L^\infty(\Omega)' \neq L^1(\Omega)$, $L^\infty(\Omega)' = \mathcal{M}(\Omega) =$ Radon Measures

Remark E.2.

Fact: $L^\infty(\Omega)' \subset L^1(\Omega)$, and the inclusion is strict

Example E.3.

Let ϕ_0 be a continuous linear functional on $C_0(\mathbb{R}^d)$ with

$$\phi_0(f) = f(0) \;\forall\; f \in C_0(\mathbb{R}^d) \tag{E.1}$$

By the Hahn-Banach Theorem, we can extend ϕ_0 to a linear functional ϕ on $L^\infty(\mathbb{R}^d)$ such that $\phi(f) = f(0) \;\forall\; f \in C_0(\mathbb{R}^d)$. Suppose (for contradiction) that there exists $u \in L^1(\mathbb{R}^d)$ such that

$$\phi(f) = \int_{\mathbb{R}^d} uf\,dx \;\forall\; f \in L^\infty(\mathbb{R}^d)$$

Then $\int_{\mathbb{R}^d} uf\,dx = f(0) = 0 \;\forall\; f \in C_0(\mathbb{R}^d)$ such that $f(0) = 0$. Then $u = 0$ a.e. on $\mathbb{R}^d \setminus \{0\}$, which implies that $u = 0$ on $\mathbb{R}^d$, and thus $\int_{\mathbb{R}^d} uf\,dx = 0 \;\forall\; f \in L^\infty(\mathbb{R}^d)$, which contradicts (E.1).

Definition E.4. ***Weak Convergence***

For $1 \leq p < \infty$, f_n *converges weakly* to f in L^p, written $f_n \rightharpoonup f$, if

$$\int_\Omega f_n g \, dx \to \int_\Omega f g \, dx \;\forall\; g \in L^q(\Omega)$$

Definition E.5. ***Weak-$*$ Convergence***

(Recall: $L^1(\Omega)' = L^\infty(\Omega)$, but $L^\infty(\Omega)' \neq L^1(\Omega)$)

f_n *converges weak-$*$* to f in $L^\infty(\Omega)$, written $f_n \overset{*}{\rightharpoonup} f$, if

$$\int_\Omega f_n g \, dx \to \int_\Omega f g \, dx \;\forall\; g \in L^1(\Omega)$$

Problem E.6.

Problem D.2 revisited

Let $u \in L^\infty(\mathbb{R}^d)$, $\|z_n\|_{L^\infty(\mathbb{R}^d)} \leq 1$, and $z_n(x) \to z(x)$ a.e. Let $v_n = \rho_{1/n} * (z_n u)$ and $v = zu$. We showed that $v_n \to v$ in $L^1_{\text{loc}}(\mathbb{R}^d)$. Now show that $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(\mathbb{R}^d)$.

Hint: Let $\underline{\rho}(x) = \rho(-x)$ in $\mathbb{R}^d$. Then $\int_{\mathbb{R}^d} (\rho * f)\phi \, dx = \int_{\mathbb{R}^d} f(\rho * \phi) \, dx$

Problem E.7.

Let $U \in L^2(\mathbb{R})$ and let $u_n(x) = U(x+n)$. Show $u_n \rightharpoonup 0$ in $L^2(\mathbb{R})$. In other words, we want:

$$\int_\mathbb{R} u_n(x)\phi(x) \, dx \to 0 \text{ as } n \to \infty \;\forall\; \sout{\phi \in L^2(\mathbb{R})} \text{ simple functions with compact support}$$

Lemma E.8.

If $f_n \to f$ in L^p then

1. $\|f\|_{L^p} \leq \liminf_{n\to\infty} \|f_n\|_{L^p}$
2. f_n is bounded in L^p

Theorem E.9.

If $1 < p < \infty$ and $\|f_n\|_{L^p(\Omega)} \leq M$, then there exists a subsequence that converges weakly in L^p, $f_{n_k} \rightharpoonup f$ in $L^p(\Omega)$.

If $p = \infty$ and $\|f_n\|_{L^\infty(\Omega)} \leq M$, then there exists a subsequence that converges weak-$*$ in $L^\infty(\Omega)$, $f_{n_k} \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.

Theorem E.10. ***Young's Inequality***

If $f \in L^1$ and $g \in L^p$, then $f * g \in L^p$ and

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}$$

More generally,

$$\|f * g\|_{L^r} \leq \|f\|_{L^q}\|g\|_{L^p} \quad \text{where } \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

F 4-6-11 (Sobolev Spaces)

Remark F.1.

1-D:

$$\frac{d^2u}{dx^2} = f \quad \text{in } (0,1)$$
$$u(0) = u(1) = 0$$
$$f \in C^0(0,1)$$

We know by definition that if $u \in C^2(0,1)$ then $f = \frac{d^2u}{dx^2} \in C^0(0,1)$.
Question: Given $f \in C^0(0,1)$, is $u \in C^2(0,1)$? Yes, by the Fundamental Theorem of Calculus.

2-D:

$$\nabla u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \text{ on } \partial\Omega$$

1. If $u \in C^2(\Omega)$ then $f \in C^0(\Omega)$
2. Let $u = \nabla^{-1} f$. If $f \in C^0(\Omega)$, is $u \in C^2(\Omega)$? No.

$C^k(\Omega)$ is not a good functional framework.

Definition F.2. ***Weak 1st Derivative in 1-D***

For $u \in L^1_{\text{loc}}(\Omega)$, $\Omega \subset \mathbb{R}$ open, if there exists $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega u(x) \frac{d\phi}{dx}\, dx = -\int_\Omega v(x)\phi(x)\, dx$$

then v is the *weak 1st derivative* of u.

Definition F.3. ***Sobolev Space*** $W^{1,p}(\Omega)$

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; 1) \text{ weak derivative } v \text{ exists}, \; 2) \; v \in L^p(\Omega) \right\}$$

Notation:
We denote $\frac{du}{dx} = v$, and in 1-D $u' = v$. Thus, $W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; u' \in L^p(\Omega) \right\}$.

Definition F.4. ***Norm on*** $W^{1,p}(\Omega)$

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|u'\|^p_{L^p(\Omega)}\right)^{1/p}$$

Definition F.5. ***Topology of*** $C^\infty(\Omega)$

$\phi_n \to \phi$ in $C^\infty(\Omega) = \mathcal{D}(\Omega)$ if

1. spt $(\phi_n - \phi) \subset K \subset\subset \Omega \; \forall \; n$
2. $\mathcal{D}^\alpha \phi_n \to \mathcal{D}^\alpha \phi$ uniformly on k

Remark F.6.

Fact: $C^\infty(\Omega)$ is not normable. The dual space $\mathcal{D}'(\Omega)$ is even worse.

Example F.7.

Is $u(x) = |x|$ for $\Omega = (-1, 1)$ in $W^{1,p}(-1, 1)$?
Step 1:

$$\begin{aligned}
\int_\Omega v(x)\phi(x)\,dx &= -\int_\Omega |x|\frac{d\phi}{dx}\,dx \\
&= -\int_{-1}^{0} -x\frac{d\phi}{dx}\,dx - \int_0^1 x\frac{d\phi}{dx}\,dx \\
&= -\int_{-1}^{0} \phi(x)\,dx + \int_0^1 \phi(x)\,dx \\
v(x) &= \frac{x}{|x|}
\end{aligned}$$

Step 2: Yes, $u \in W^{1,p}(\Omega)$.

Definition F.8. ***Weak Derivative***

Given: $u \in L^1_{\text{loc}}(\Omega),\ \Omega \subset \mathbb{R}^d,\ \alpha$ is a multi-index.
If there exists $v^{(\alpha)} \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega u(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v^{(\alpha)}(x)\phi(x)\, dx \quad \forall\ \phi \in C_0^\infty(\Omega)$$

then $v^{(\alpha)}$ is the α-th derivative of u.

Notation: Denote $D^\alpha u = v^{(\alpha)}$.

Definition F.9. $W^{k,p}(\Omega)$

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega)\ \middle|\ 1)\ v^{(\alpha)} \text{ exists in } L^1_{\text{loc}},\ 2)\ v^{(\alpha)} \in L^p(\Omega)\ \forall\ |\alpha| \le K \right\}$$

G 4-8-11

Definition G.1. ***Norm***

For every $u \in W^{k,p}(\Omega)$,

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)} \right)^{1/p}, \; 1 \le p \le \infty$$

Theorem G.2.

$W^{k,p}$ is a Banach space.

Proof. Consider $W^{1,p}$. Let (u_n) be any Cauchy sequence in $W^{1,p}$. So $u_n \to u$ in $L^p(\Omega)$ and the weak derivative $Du_n \to v$ in $L^p(\Omega)$. We want to show that v is the weak derivative of u, i.e. that

$$\int_\Omega u D\phi \, dx = - \int_\Omega \phi \, dx$$

We know that this is true by the Dominated Convergence Theorem. □

Lemma G.3.

If $u_n \to u$ in L^p strongly, then $u_n \rightharpoonup u$ in $L^p(\Omega)$.

Proof. Hölder's inequality. □

Definition G.4. ***Convergence in a Sobolev Space***

We say that $u_n \to u$ in $W^{k,p}(\Omega)$ if $\|u_n - u\|_{W^{k,p}(\Omega)} \to 0$.

We'll see that

$$W^{1,1} = \{\text{absolutely continuous functions}\}$$
$$W^{1,\infty} = \{\text{Lipschitz functions (uniformly continuous)}\}$$

Remark G.5. ***Notation:*** $H^k(\Omega)$

For $p = 2$, we say that $K^k(\Omega) = W^{k,2}(\Omega)$, with $k = 1$ or 2.

Consider $H^1(\Omega)$. If $k = \frac{d}{2}$, where $d = \dim(\Omega)$, then $f \in H^k(\Omega) \Rightarrow f$ is continuous.

Example G.6.

(2-D) Let $u(x) = |x|^{1/2}$ and $\Omega = B(0,1)$. For which values of p is u in $W^{1,p}(\Omega)$?

Step 1: i. $\|u\|_{L^p(\Omega)} < \infty$, ii. u has weak derivative v, iii. $v \in L^p(\Omega)$, $\|v\|_{L^p(0,1)} < \infty$

$$\int_\Omega |u|^p \, dx = \int_{B(0,1)} |x|^{p/2} \, dx < \infty \ \forall \ p \in [1, \infty)$$

Step 2:

$$\frac{\partial u}{\partial x_i} = \frac{1}{2}|x|^{-1/2} \frac{\partial}{\partial x_i}|x| = \frac{1}{2} \frac{x_i}{|x|^{3/2}} \quad \text{for } x \neq 0$$

This is true because

$$|x| = \left(\sum_{i=1}^{2} x_i x_i \right)^{1/2} \Rightarrow |x| = \frac{1}{2} \left(\sum_{i=1}^{2} x_i x_i \right)^{-1/2} \cdot 2x_i = \frac{x_i}{|x|}$$

Guess that $v(x) = \frac{1}{2} \cdot \frac{x_i}{|x|^{3/2}}$. Goal: prove that $\int_\Omega u \frac{\partial \phi}{\partial x_i} \, dx = -\int_\Omega v(x)\phi(x) \, dx \ \forall \ \phi \in C_0^\infty(\Omega)$.
Note that the weak derivative in multiple dimensions is synonymous with the weak gradient.
Remove a ball $B(0,\delta))$ from Ω to get the region $\Omega_\delta = B(0,1) - B(0,\delta)$. Let n_i denote the ith component of the unit normal on the boundary. Then by Integration By Parts / The Divergence Theorem, we get

$$\begin{aligned}
\int_{\Omega_\delta} u(x) \frac{\partial \phi}{\partial x_i} \, dx &= \int_{\partial \Omega_\delta = \partial B(0,\delta)} u(x)\phi(x) n_i \, dS - \int_{\Omega_\delta} \frac{\partial u}{\partial x_i} \phi(x) \, dx \\
&= \int_0^{2\pi} \delta^{1/2} \phi(x) \underbrace{n_i}_{|n_i|=1} \delta \, d\theta - \frac{1}{2} \int_0^{2\pi} \int_0^1 \frac{x_i}{|x|^{3/2}} \phi(x) \, dx \\
&\leq \underbrace{\delta^{3/2} \int_0^{2\pi} |\phi(x) \, d\theta}_{\to 0 \text{ as } \delta \to 0} + \underbrace{\frac{1}{2} \int_0^{2\pi} \int_0^1 \mathbf{1}_{(\delta,1)} |x|^{-1/2} |\phi(x)| \, dx}_{\text{see next line}}
\end{aligned}$$

$$\frac{1}{2} \int_0^{2\pi} \int_0^1 \mathbf{1}_{(\delta,1)} |x|^{-1/2} |\phi(x)| \, dx = \frac{1}{2} \int_0^{2\pi} \int_0^1 \mathbf{1}_{(\delta,1)} r^{-1/2} \underbrace{|\phi(r,\theta)|}_{\substack{L^1 \\ \text{dominating} \\ \text{function}}} r \, dr \, d\theta$$

By the Dominated Convergence Theorem we can pass to the limit as $\delta \to 0$, and this second term goes to $-\int_\Omega v(x)\phi(x) \, dx$. Thus, $v(x) = \frac{1}{2} \cdot \frac{x_i}{|x|^{3/2}}$.
For what p is $v \in L^p$, i.e. when is $\int_\Omega |x|^{-p/2} \, dx < \infty$?
Answer: switch to polar coordinates and get that $p < 4$ (Shkoller thinks)

Remark G.7. ***Sobolev Embedding and the Fundamental Theorem of Calculus***

$$\max |u(x)| \leq C\|u\|_{W^{k,p}(\Omega)}, \ \forall \ u \in C_0^\infty(\Omega) \text{ and } x \in \text{spt } (u), \ \Omega \subset \mathbb{R}^2, \ kp > 2$$

Dimension $d = 2$, so suppose $p = 2 \Rightarrow k > 1$. But if $p = s, \ k > 2/3 \Rightarrow k = 1$ "works," and $W^{1,3}$ now consists of continuous functions. Choose a coordinate system such that $x = 0$.

$$u(r) = -\int_r^1 \partial_s u(s, \theta)\, ds$$

We need to address issues:

- Integration by parts
- Cut-off functions

H 4-11-11

Theorem H.1. ***Sobolev Embedding Theorem (2-D Version)***

$$\max_{x \in \text{spt }(u)} |u(x)| \leq C\|u\|_{W^{k,p}(\Omega)} \; \forall \; u \in C_0^k(\Omega), \; kp > 2$$

where C = generic constant = $C(k, p, \Omega, d)$.

Proof.

$$|u(x)| \leq C\|u\|_{W^{k,p}(\Omega)} \; \forall \; x \in \text{spt }(u)$$

$$\text{Shift } x \text{ to } 0: \quad |u(0)| \leq C(r) \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}$$

By the Fundamental Theorem of Calculus,

$$u(x) - u(0) = \int_0^x \frac{\partial u}{\partial r}(r, \theta)\, dr$$

Choose $\psi \in C_0^\infty(B(0,1))$ such that $\psi \equiv 1$ on $B\left(0, \frac{1}{2}\right)$, $\psi \equiv 0 \; \forall \; |x| \geq \frac{3}{4}$. Replace $u \mapsto \psi u$.

$$\begin{aligned}
-\psi u(0) = -u(0) &= \int_0^1 \frac{\partial}{\partial r}(\psi u)\, dr \\
u(0) &= -\int_0^1 \frac{\partial}{\partial r}(\psi u)\, dr \\
&= -\int_0^1 \frac{\partial}{\partial r}(r) \frac{\partial}{\partial r}(\psi u)\, dr \\
&\overset{\text{IBP}}{=} \int_0^1 f \frac{\partial^2}{\partial r^2}(\psi u)\, dr - \cancel{r\psi u}\Big|_0^{\cancel{1}} \\
&= C_k \int_0^1 r^{k-1} \frac{\partial^k}{\partial r^k}(\psi u)\, dr
\end{aligned} \tag{H.1}$$

We are missing 3 things: 1) lower order derivatives, 2) integral over 2-D region, 3) powers of p.

$$\begin{aligned}
&x = r\cos\theta, \quad y = r\sin\theta \\
&\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} = A(\theta) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = A(\theta)D, \quad A \in C^\infty(\theta), \quad D = \text{gradient} \\
&\frac{\partial^2}{\partial r^2} = A(\theta)D^2 \Rightarrow \frac{\partial^k}{\partial r^k} = \sum_{|\alpha| \leq k} A^\alpha(\theta) D^\alpha \quad \text{(chain rule for smooth terms)}
\end{aligned}$$

Then continuing from (H.1), we get that

$$
\begin{aligned}
u(0) &= C_k \int_0^1 r^{k-1} \sum_{|\alpha|\le k} A^\alpha(\theta) D^\alpha(\psi u)\, dr \\
&= C_k \int_0^{2\pi}\int_0^1 r^{k-2} \sum_{|\alpha|\le k} A^\alpha(\theta) D^\alpha(\psi u)\, \underbrace{r\, dr\, d\theta}_{\substack{\text{Lebesgue}\\ \text{measure}}} && \text{(integrated over } \theta \text{ from 0 to } 2\pi) \\
&\le C\left(\int_{B(0,1)} r^{\frac{p(k-2)}{p-1}} r\, dr\, d\theta\right)^{\frac{p-1}{p}} \left(\sum_{|\alpha|\le k}\int_{B(0,1)} |D^\alpha(\psi u)|^p\, dx\right)^{\frac{1}{p}} && \text{(Hölder's Inequality)}
\end{aligned}
$$

The first integral is legitimate when $\frac{p(k-2)}{p-1} + 1 > -1 \Rightarrow kp > 2$. □

Remark H.2.

The Poisson kernel gives us the solution $u = P_r * g$ to

$$
\begin{aligned}
\Delta u &= 0 \quad \text{in } B(0,1) \\
u &= g \quad \text{on } \partial B(0,1)
\end{aligned}
$$

But what if we have an irregular domain?

Motivation:
Let $v \in C_0^\infty(\Omega)$. Then we have

$$
\begin{aligned}
0 &= -\int_\Omega \Delta u v\, dx = -\int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_i} v\, dx \\
&= -\int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_i} v\, dx \\
&= -\int_\Omega \left[\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i} v\right) - \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right] dx \\
&= \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx - \int_\Omega \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i} v\right) dx \\
&= \int_\Omega Du\cdot Dv\, dx - \underbrace{\int_\Omega \operatorname{div}(vDu)\, dx}_{\int_{\partial\Omega} vDu\cdot n\, dS}
\end{aligned}
\qquad = \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial x_i} v n_i\, dS
$$

where n_i is the ith component of the outward unit normal and $\frac{\partial u}{\partial n} = Du \cdot n$. Thus, we have

Classical Form:

$$
\begin{aligned}
\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

New Form:

$$
\int_\Omega Du\cdot Dv\, dx = \underbrace{\cancel{\int_{\partial\Omega} \frac{\partial u}{\partial n} v\, dS}}_{\text{since } v=0 \text{ on } \partial\Omega} = 0 \ \forall\ v \in C_0^\infty(\Omega)
$$

$$
\int_\Omega Du\cdot Dv\, dx = 0 \qquad \forall \qquad \underbrace{v \in C_0^\infty(\Omega)}_{v\in H^1(\Omega),\ v=0 \text{ on } \partial\Omega}
$$

Remark H.3. ***Notation: Einstein Summation***

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_i} = \frac{\partial^2 u}{\partial x_i \partial x_i}$$

Remark H.4.

Fact: $C_0^\infty(\Omega)$ is dense in a certain subspace of $H^1(\Omega)$.

I 4-13-11

Theorem I.1. ***Morrey's Inequality***

Given: $y \in B(x,r) \subset \mathbb{R}^d,\ p > d$. Then

$$|u(x) - u(y)| \leq Cr^{1-d/p} \|Du\|_{L^p(B(x,2r))} \qquad \forall\ u \in \underbrace{C^\infty(\overline{B(x,2r)})}_{\text{or } C^1}$$

Corollary I.2. ***Sobolev Embedding*** ***($k=1$)***

$W^{1,p} \hookrightarrow C^{0,1-d/p}(\overline{\Omega})$
There exists $C > 0$ such that $\|u\|_{C^{0,1-d/p}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)} \qquad \forall\ u \in W^{1,p}(\Omega)$.

Definition I.3. $C^{0,\gamma}(\overline{\Omega})$

$C^{0,\gamma}(\overline{\Omega})$ = Hölder space with the norm given by

$$\|u\|_{C^{0,\gamma}} = \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}$$
$$[u]_{C^{0,\gamma}(\overline{\Omega})} = \max \frac{|u(x) - u(y)|}{|x-y|^\gamma}$$

this interpolates between C^0 and C^1.

Remark I.4. ***Notation:*** $⨍$

$⨍_\Omega f(x)\,dx = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$ = average value of f over Ω

Lemma I.5.

$$⨍_{B(x,r)} |u(x) - u(y)|\,dy \leq C \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{d-1}}\,dy \qquad y \in B(x,r)$$

Proof. (2-D)

$$y = x + se^{i\theta}, \qquad s \in (0,r), \qquad e^{i\theta} \in S^1 = \partial B(0,1)$$

$$
\begin{aligned}
u(y) - u(x) &= u(x + se^{i\theta}) - u(x) \\
&= \int_0^s \partial_\tau (x + \tau e^{i\theta})\, d\tau && \text{FTOC} \\
&= \int_0^s Du(x + \tau e^{i\theta}) e^{i\theta}\, d\tau && \text{chain rule} \\
|u(y) - u(x)| &\leq \int_0^s |Du(x + \tau e^{i\theta})|\, d\tau \\
\int_0^{2\pi} |u(y) - u(x)|\, d\theta &\leq \int_0^{2\pi} \int_0^s |Du(x + \tau e^{i\theta})|\, d\tau\, d\theta \\
&\leq \int_0^{2\pi} \int_0^s \frac{|Du(x + \tau e^{i\theta})|}{\tau} \underbrace{\tau\, d\tau\, d\theta}_{\text{measure}} \\
&\leq \int_0^{2\pi} \int_0^s \frac{|Du(y)|}{|x - y|}\, dy \\
&\leq \int_{B(x,r)} \frac{|Du(y)|}{|y - x|}\, dy \\
\int_0^r \int_0^{2\pi} |u(y) - u(x)|\, d\theta\, d\tilde{r} &\leq \int_0^r \int_{B(x,r)} \frac{|Du(y)|}{|y - x|}\, dy\, d\tilde{r} \\
⨍_{B(x,r)} |u(y) - u(x)|\, dy &\leq C \left(\int_{B(x,r)} \left(\frac{1}{s^{d-1}} \right)^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |Du|^p\, dy \right)^{\frac{1}{p}} && \text{Hölder's}
\end{aligned}
$$

Let $Z = B(x,r) \cap B(y,r)$. Then

$$
|u(x) - u(y)| \leq |u(x) - u(z)| + |u(z) - u(y)|
$$

Integrating this over Z gives

$$
\begin{aligned}
|Z||u(x) - u(y)| &\leq \int_Z |u(x) - u(z)|\, dz + \int_Z |u(z) - u(y)|\, dz \\
|u(x) - u(y)| &\leq ⨍_Z |u(x) - u(z)|\, dz + ⨍_Z |u(z) - u(y)|\, dz \\
&\leq \int_{B(x,2r)} |u(x) - u(z)|\, dz + \int_{B(x,2r)} |u(z) - u(y)|\, dz
\end{aligned}
$$

□

Theorem I.6. ***Interior Approximation***

$C^\infty(\Omega_\epsilon)$ is dense in $W^{k,p}(\Omega)$, meaning that for every $u \in W^{k,p}(\Omega)$ there exists $u^\epsilon \in C^\infty(\Omega_\epsilon)$ such that

$$
\begin{aligned}
&u^\epsilon \to u \text{ in } W^{k,p}_{\text{loc}}(\Omega) \\
&u^\epsilon \to u \text{ in } W^{k,p}(\tilde{\Omega})\ \forall\ \tilde{\Omega} \subset\subset \Omega
\end{aligned}
$$

Remark I.7.

Suppose that $v^{(\alpha)}$ is the αth derivative of u $\forall$ $|\alpha| \leq k$. We want to show:

$$D^\alpha u^\epsilon \to v^{(\alpha)} \text{ as } \epsilon \searrow 0 \text{ in } L^p_{\text{loc}}(\Omega)$$

Why is u^ϵ smooth?
Let $u^\epsilon = \rho_\epsilon * u$. This is smooth by the LDCT.

$$\begin{aligned} D^\alpha \int_{\Omega_\epsilon} \rho_\epsilon(x-y)u(y)\,dy &= \int_{\Omega_\epsilon} D^\alpha_y \rho_\epsilon(x-y)u(y)\,dy \\ &= (-1)^{|\alpha|} \int_{\Omega_\epsilon} D^\alpha_y \rho_\epsilon(x-y)u(y)\,dy \\ &= (-1)^{|\alpha|} \int_{\Omega_\epsilon} \rho_\epsilon(x-y)v^{(\alpha)}(y)\,dy \end{aligned}$$

J 4-15-11

Lemma J.1. ***Review from Last Time***

Let $y \in B(x,r)$. Then

$$⨍_{B(x,r)} |u(y)-u(x)|\,dy \le C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{d-1}}\,dy$$

Idea: $y = x + sw,\ w \in S^{d-1}$

$$\int_0^r \int_{S^{d-1}} |u(x+sw)-u(x)| \underbrace{dw}_{xs^{d-1}\,ds} \le \int_0^r \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{d-1}}\,dy\ s^{d-1}\,ds$$

Theorem J.2. ***Review from Last Time***

$$|u(y)-u(x)| \le Cr^{1-d/p}\|Du\|_{L^p(B(x,2r))} \quad \forall\ u \in C^1$$

Morrey's inequality comes from Hölder's Inequality:

$$\left(\int_B \left(\frac{1}{s^{d-1}}\right)^{\frac{p}{p-1}} s^{d-1}\,ds\,dw\right)^{\frac{p-1}{p}} \left(\int_B |Du|^p\,dx\right)^{\frac{1}{p}}$$

Integrability determines the embedding (integrability requires $p > d$).

Theorem J.3. ***Sobolev Embedding Theorem ($k = 1$)***

$p > d$, $W^{1,p} \hookrightarrow C^{0,1-d/p}$

$$\|u\|_{C^{0,1-d/p}(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\mathbb{R}^d)} \qquad \forall\ u \in W^{1,p}(\mathbb{R}^d)$$

Example: $d = 1$
$H^1 \hookrightarrow C^{0,1/2}$ ($\frac{1}{2}$ derivative gain)

Remark J.4. ***Density***

For Ω bounded, $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for $1 \le p < \infty$.

$\mathbb{R}^d$: $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.

Proof. (Sobolev Embedding Theorem, $k = 1$) Suppose we are working with $C_0^1(\mathbb{R}^d)$. Morrey's Inequality gives us that

$$\frac{|u(y)-u(x)|}{r^{1-d/p}} \le C\|Du\|_{L^p(B(x,2r))}$$

So it suffices to prove that $|u(x)| \le C\|u\|_{W^{1,p}(\mathbb{R}^d)}$.
Recall: by definition, $\|u\|_{C^{0,1-d/p}(\mathbb{R}^d)} = \max|u(x)| + \max \frac{|u(y)-u(x)|}{|y-x|^{1-d/p}}$.

$$\begin{aligned}|u(x)| &\le \fint_{B(x,1)} |u(y)-u(x)|\,dy + \fint_{B(x,1)} |u(y)|\,dy \\ &\le C\fint_{B(x,1)} \frac{|Du(y)|}{|y-x|^{d-1}}\,dy + C\|u\|_{L^p} \\ &\le C\|u\|_{W^{1,p}(\mathbb{R}^d)} \quad \forall\, u \in C_0^1(\mathbb{R}^d),\ x \in \text{spt }(u)\end{aligned}$$

□

Remark J.5.

Suppose there exists $u_j \in C_0^\infty(\mathbb{R}^d)$ such that $u_j \to U$ in $C^{0,1-d/p}$. Then $U = u$ a.e., and

$$\|u_j\|_{C^{0,1-d/p}} \le C\|u_j\|_{W^{k,p}}$$
$$\|U\|_{C^{0,1-d/p}} \le C\|U\|_{W^{k,p}}$$

Corollary J.6.

If $d < p$ then the weak derivative of $u \in W^{1,p}$ is equal to the classical derivative a.e.

Theorem J.7. ***Gagliardo-Nirenberg***

Suppose $d > p \ge 1$. Let $p^* = \frac{dp}{d-p}$. Then

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \le C\|Du\|_{L^p(\mathbb{R}^d)} \qquad \forall\, u \in W^{1,p}$$

(For example, if we have $d = 2$ and $p = 1$ then $p^* = 2$ and $\|u\|_{L^2} \le C\|Du\|_{L^1}$)

Problem J.8. ***Hardy's Inequality***

Suppose $\Omega = (0,1)$, $u \in H^1$, $u(0) = 0$. Then $\frac{u}{x} \in L^2(0,1)$, and

$$\left\|\frac{u}{x}\right\|_{L^2(0,1)} \le C\|u\|_{H^1}$$

Problem J.9. ***Hardy's Inequality (Simple Version)***

Suppose $\Omega = (0,1),\ u \in H^1,\ u(0) = u(1) = 0.$ Prove

$$\left\|\frac{u}{x}\right\|_{L^2} \le 2\|u'\|_{L^2}$$

(HINT: Let $v = \frac{u}{x}$ so that $u = xv$.)

WANT: $\|v\|_{L^2} \le C\|(xv)'\|_{L^2}$.

$$\begin{aligned}(xv)' =& xv' + v \in L^2\\ & xv' + v = 0 \quad \Rightarrow \quad v = \frac{1}{x} \notin L^2\end{aligned}$$

K 4-18-11

Theorem K.1. ***Hardy's Inequality (from last time)***

Let $u \in H^1(0,1),\ u(0) = u(1) = 0\ \big(u \in H_0^1(0,1)\big)$.
Then $\frac{u}{x} \in L^2(0,1)$ and

$$\left\|\frac{u}{x}\right\|_{L^2(0,1)} \leq C\|u\|_{H^1(0,1)}$$

Recall: $\|u\|_{H^1(0,1)}^2 = \|u\|_{L^2(0,1)}^2 + \|u'\|_{L^2(0,1)}^2$. Thus, we need to prove that

$$\left\|\frac{u}{x}\right\|_{L^2(0,1)} \leq c\|u'\|_{L^2(0,1)}$$

Proof. Let $v = \frac{u}{x} \Rightarrow u = xv$. Want: $\|v\|_{L^2} \leq C\|(xv)'\|_{L^2} = C\|xv' + v\|_{L^2}$.
Formal computation:

$$\begin{aligned}
\|xv' + v\|_{L^2}^2 &= \big\langle xv' + v, xv' + v\big\rangle_{L^2} \\
&= \int_0^1 (x^2 v'^2 + \underbrace{2xv'v}_{\substack{\text{cross-term} \\ \text{CT}}} + v^2)\, dx \\
\text{CT} &= \int_0^1 2x \frac{dv}{dx} v\, dx \\
&= \int_0^1 \frac{d}{dx}|v|^2\, dx = -\int_0^1 |v|^2\, dx
\end{aligned}$$

$$\|xv' + v\|_{L^2}^2 = \|xv'\|_{L^2}^2$$

But how do we make this rigorous?
Start with smooth functions and show that

$$\begin{aligned}
\|v\|_{L^2} &\leq C\|(xv)'\|_{L^2} \quad \forall\ u \text{ smooth} \\
\left\|\frac{u}{x}\right\|_{L^2} &\leq C\|u'\|_{L^2} \quad \forall\ u \text{ smooth},\ C_0^\infty(0,1)
\end{aligned}$$

Then $v \in C_0^\infty$ and $\lim_{x \searrow 0} xv^2 = 0$. Using this dense subset of smooth functions rules out singular behavior. □

Remark K.2. ***Sobolev Embedding (Scaling)***

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\left(\|u\|_{L^p}(\mathbb{R}^n) + \|Du\|_{L^p(\mathbb{R}^n)}\right) \quad \forall\ u \in W^{1,p}(\mathbb{R}^n),\ p > n$$

Let $v(x) = u\left(\frac{x}{\lambda}\right)$. Then $v \in W^{1,p}$ and

$$\|v\|_{L^\infty(\mathbb{R}^n)} \leq C\left(\|v\|_{L^p} + \|Dv\|_{L^p}\right) \tag{K.1}$$

Compute $\|v\|_{L^p}$ and $\|Dv\|_{L^p}$:

$$\int_{\mathbb{R}^n} |v(x)|^p\,dx = \int_{\mathbb{R}^n} \left|u\left(\frac{x}{\lambda}\right)\right|^p\,dx = \lambda^n \int_{\mathbb{R}^n} |v(y)|^p\,dy$$
$$\int_{\mathbb{R}^n} |Dv(x)|^p\,dx = \int_{\mathbb{R}^n} \left|Du\left(\frac{x}{\lambda}\right)\right|^p\,dx = \lambda^{n-p} \int_{\mathbb{R}^n} |Du(y)|\,dy$$

where the λ^n term in the first equation is due to the Jacobian. Plugging these into (K.1) yields

$$\|u\|_{L^\infty} \leq C\left(\lambda^{\frac{n}{p}}\|u\|_{L^p} + \lambda^{\frac{n-p}{p}}\|Du\|_{L^p}\right) \tag{K.2}$$

Minimize the right hand side by taking a derivative with respect to λ:

$$\begin{aligned} 0 &= \frac{n}{p}\lambda^{\frac{n}{p}-1}\|u\|_{L^p} + \frac{n-p}{p}\lambda^{\frac{n}{p}-1-1}\|Du\|_{L^p} \\ &= \lambda^{\frac{n}{p}-1}\left[\frac{n}{p}\|u\|_{L^p} + \lambda^{-1}\frac{n-p}{p}\|Du\|_{L^p}\right] \\ \lambda &= \frac{\|Du\|_{L^p}}{\|u\|_{L^p}}C(n,p) \end{aligned}$$

Plugging this into (K.2) yields

$$\begin{aligned} \|u\|_{L^\infty} &\leq C\left(\frac{\|Du\|_{L^p}^{n/p}}{\|u\|_{L^p}^{n/p}}\|u\|_{L^p} + \frac{\|Du\|_{L^p}^{\frac{n-p}{p}+1}}{\|u\|_{L^p}^{\frac{n-p}{p}}}\right) \\ &\leq C\|Du\|_{L^p}^{\frac{n}{p}}\|u\|_{L^p}^{\frac{p-n}{p}}, \quad n < p \end{aligned}$$

Note: $\frac{n}{p} + \frac{p-n}{p} = 1$.
This result is called an *interpolation identity*.

Example K.3. ***Green's Function***

Consider $-\Delta u = f$ in $\mathbb{R}^n$.
A *Green's function* $G(x-y)$ satisfies $-\Delta G = \delta$ in $\mathcal{D}'(\mathbb{R}^n)$.
The solution is given by $u = G * f$, and G is called the *fundamental solution*.

$$\begin{aligned} \text{2-D}: &\ G = C \log|x| \\ \text{3-D}: &\ G = C \cdot \frac{1}{|x|} \end{aligned}$$

Note that these functions are smooth everywhere except the origin; they are very singular at the origin.
Suppose $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta \equiv 1$ in a neighborhood of 0.

$$\begin{aligned} F &= \theta G \\ -\Delta F &= \delta - \psi,\ \psi \in C_0^\infty(\mathbb{R}^n) \\ u &= -u * \Delta F + u^* \psi = Du * DF + u * \psi \end{aligned}$$

Young's Inequality:

$$\|u\|_{L^\infty} \le C\left(\|Du\|_{L^p}\|DF\|_{L^q} + \|u\|_{L^p}\|\psi\|_{L^q}\right)$$

$DF \in L^q,\ p > n$ and $\psi \in L^q$.

L 4-20-11

Remark L.1.

$p > n \;\Rightarrow$ Classical differentiability
$p < n \;\Rightarrow$ Gagliardo-Nirenberg

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall\ u \in C_0^1(\mathbb{R}^n)$$

where $p^* = \frac{np}{n-p}$, $1 \le p < n$.

Scaling Argument
If this holds for $u(x)$, $x \in \mathbb{R}^n$, then it holds for $v(x) = \frac{u(x)}{\lambda}$, $\lambda \in \mathbb{R}$.

$$\|v\|_{L^{p^*}(\mathbb{R}^n)} = \lambda^{n/p^*}\|u\|_{L^{p^*}(\mathbb{R}^n)}$$
$$\|Dv\|_{L^p(\mathbb{R}^n)} = \lambda^{\frac{n-p}{p}}\|Du\|_{L^p(\mathbb{R}^n)}$$
$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\lambda^{\left(\frac{n-p}{p}-\frac{n}{p^*}\right)}\|Du\|_{L^p(\mathbb{R}^n)}$$

we must have that
$$\frac{n-p}{p} = \frac{n}{p^*} \quad \Rightarrow \quad p^* = \frac{np}{n-p}$$

Example L.2.

$n = 2,\ 1 \le p < 2$

$p = 1$	$p^* = 2$	$\|u\|_{L^2(\mathbb{R}^2)} \le C\|Du\|_{L^1(\mathbb{R}^2)}$
$p = \frac{3}{2}$	$p^* = 6$	$\|u\|_{L^6(\mathbb{R}^2)} \le C\|Du\|_{L^{3/2}(\mathbb{R}^2)}$
$p = \frac{199}{100}$	$p^* = 398$	$\|u\|_{L^{398}(\mathbb{R}^2)} \le C\|Du\|_{L^{199/100}(\mathbb{R}^2)}$
$p \nearrow 2$	$p^* \to \infty$	$\|u\|_{L^\infty(\mathbb{R}^2)} \not\le C\|Du\|_{L^2(\mathbb{R}^2)} \not\le C\|u\|_{H^1}$

Theorem L.3.

$(n = 2 = p) \qquad \forall\ q \in [1,\infty)$:

$$\|u\|_{L^q(\mathbb{R}^2)} \le C\sqrt{q}\|u\|_{H^1(\mathbb{R}^2)} \qquad \forall\ u \in C_0^1(\mathbb{R}^2)$$

Proof of Gagliardo-Nirenberg $(n = 2)$
Step 1: $p = 1,\ p^* = 2$, prove $\|u\|_{L^2} \le C\|Du\|_{L^1}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x_1,x_2)|^2\,dx_1\,dx_2 \le C\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2)|\,dx_1\,dx_2\right)^2$$
$$\le C\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2)|\,dx_1\,dx_2\right)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2)|\,dx_1\,dx_2\right)$$

We want to apply the Fundamental Theorem of Calculus.

$$
\begin{aligned}
u(x_1, x_2) &= \int_{-\infty}^{x_1} \partial_1 u(y_1, x_2)\, dy_1 = \int_{-\infty}^{x_2} \partial_2 u(x_1, y_2)\, dy_2 \\
|u(x_1, x_2)| &\leq \int_{-\infty}^{\infty} |\partial_1 u(y_1, x_2)|\, dy_1 \\
&\leq \int_{-\infty}^{\infty} |\partial_1 u(x_1, y_2)|\, dy_2 \\
|u(x_1, x_2)| &\leq \int_{-\infty}^{\infty} |\mathcal{D}u(y_1, x_2)|\, dy_1 \int_{-\infty}^{\infty} |Du(x_1, y_2)|\, dy_2 \\
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x_1, x_2)| dx_1\, dx_2 &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{D}u(y_1, x_2)|\, dy_1 \int_{-\infty}^{\infty} |Du(x_1, y_2)|\, dy_2 dx_1\, dx_2
\end{aligned}
$$

$|u| \mapsto |u|^\gamma$, plus Hölder's inequality for the general case.

Reminder: we want to prove

$$
\|u\|_{L^q(\mathbb{R}^2)} \leq C\sqrt{q}\|u\|_{H^1(\mathbb{R}^2)} \qquad \forall\ u \in C_0^1(\mathbb{R}^2)
$$

Proof. Let $r = |y - x|$. Let ψ be the same cut-off as in proof 1 of Morrey's Inequality.

$$
\begin{aligned}
|u(x)| &\leq \int_0^1 \int_0^{2\pi} \frac{|Du(y)|}{|y-x|}\, dy \\
&\leq \int_{\mathbb{R}^2} \mathbf{1}_{B(x,1)} |x-y|^{-1} |Du(y)|\, dy \\
&\leq K * Du
\end{aligned}
$$

where $K(x) = \mathbf{1}_{B(0,1)}|x|^{-1}$. We employ Young's Inequality:

$$
\begin{aligned}
&\|u\|_{L^q(\mathbb{R}^2)} \leq \|k\|_{L^k(\mathbb{R}^2)} \|Du\|_{L^2(\mathbb{R}^2))} \\
&\frac{1}{q} + 1 = \frac{1}{k} + \frac{1}{2} \quad \Rightarrow \quad k = \frac{2q}{2+q}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{2\pi}\int_0^1 \frac{1}{r^{k-1}}\, dr\, d\theta &\sim \frac{c}{2-k} r^{2-k}\Big|_0^1 & 2-k &= \frac{4}{2+q} \\
\|u\|_{L^q} &\leq c\left(\frac{q+2}{4}\right)^{1/k} \|Du\|_{L^2} & \frac{1}{2-k} &= \frac{2+q}{4} \\
&\leq c\sqrt{q}\|Du\|_{L^2} \quad \text{in the limit}
\end{aligned}
$$

□

M 4-22-11

Definition M.1. ***C^1 Domain, Localization***

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and have a C^1 boundary. This means that locally around each point, each region is dipeomorphic to $\mathbb{R}^n$. A domain is C^1 if

1. there exists an open covering on $\partial\Omega$ by K open sets $\{U_l\}_{l=1}^K$
2. For $l = 1, \ldots, k$ and $\theta_l : V_l \subset \mathbb{R}^n \to U_l$ with the following properties:
   (a) θ_l is a C^1 diffeomorphism (the map has an inverse which is also C^1).
   (b) $\theta_l(V_l^+) = U_l \cap \Omega$ (the upper half of the unit ball is mapped into Ω)
   (c) $\theta_l\big(B(0, r_l) \cap \{x_n = 0\}\big) = \partial\Omega \cap U_l$ (known as straightening the boundary)
3. there exists a collection of functions $\{\psi_l\}_{l=1}^k$ such that $\psi_l \in C_0^\infty(U_l)$, $0 \leq \psi_l \leq 1$ with $\sum_{l=1}^k \psi_l(x) = 1 \ \forall \ x \in \cup U_l$

The idea behind these partitions of unity is that if we have $u : \Omega \to \mathbb{R}$, then

$$u = u\left(\sum_{l=1}^k \psi_l(x)\right) = \sum_{l=1}^k (\psi_l u)(x).$$

This is called *localization*.

Remark M.2.

We may define $u_l = \psi_l u$ with $u = \sum u_l$. We can then remap by defining (for each l), $\mathcal{U}_l = u_l \circ \theta_l$, with $\mathcal{U}_l : V_l \to \mathbb{R}$. Then each $\mathcal{U}_l$ is zero on the boundary of these open sets. The idea now is that if we can do what is needed on a half-space, then we can do it on an arbitrary domain.

Definition M.3. $H_0^1(\Omega)$

We define $H_0^1(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the $H_1(\Omega)$ norm.

We'd like to say that $H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$. The problem is that since the boundary has measure zero, $U\big|_{\partial\Omega}$ is only defined up to equivalence classes.

Theorem M.4. ***Trace Theorem***

There exists a continuous linear operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ such that

1. $\|Tu\|_{L^2(\Omega)} \leq c\|u\|_{H^1(\Omega)}$
2. $Tu = u\big|_{\partial\Omega}$ for all $u \in C^0(\overline{\Omega}) \cap H^1(\Omega)$

Proof. Suppose first that $u \in C^1(\overline{\Omega})$. Then

$$\begin{aligned}\int_{\partial\Omega} |u|^2 \, ds &\le \int_{\partial\Omega} \sum_{l=1}^{K} |(\psi_l u)|^2 \, ds \\ &\le \sum_{l=1}^{K} \int_{\partial\Omega \cap U_l} |u_l|^2 \, ds_l\end{aligned}$$

where $u_l = \psi_l u$. We check each summand:

$$\begin{aligned}\int_{\partial\Omega \cap U_l} |u_l|^2 \, ds_l &= \int_{\theta_l(V_l \cap \{x_n = 0\})} |u_l|^2 \, ds_l \\ &= \int_{V_l \cap \{x_n=0\}} |u_l \circ \theta_l|^2 |\det D\theta_l| \, dx_1 \, \cdots \, dx_{n-1} \\ &= -\int_{V_l^+} \frac{\partial}{\partial x_n} |u_l \circ \theta_l|^2 \det D\theta_l \, dx\end{aligned}$$

where the arguments follow by localization, a change of variables and the divergence theorem. We use the product and chain rule to arrive at

$$C \int_{V_l^+} |u_l \circ \theta_l||D_l \circ \theta_l| \, dx \le \int_{U_l \cap \Omega} |u_l||Du_l| \, dx.$$

A change of variables yields the inequality in the line above. Then applying Cauchy-Schwarz gives us

$$c \int_{U_l \cap \Omega} |u_l||Du_l| \, dx \le C\|u_l\|_{L^2}^2 + \|Du_l\|_{L^2}^2.$$

We then sum over all l to yield the result. Let $\{u_j\} \in C^\infty(\overline{\Omega})$ converging in $H^1(\Omega)$ to u. Then

$$\|Tu_l - Tu_p\|_{L^2(\partial\Omega)} \le C\|u_l - u_p\|_{H^1(\Omega)}.$$

We know our sequence on the right converges, so the one on the left does as well. Hence, this defines the operator T. □

Remark M.5.

The goal behind the Trace theorem is to use

$$\int_{-\infty}^{\infty} u(x_1) \, dx_1 = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\partial u}{\partial x_2}(x_1, x_2) \, dx_1 \, dx_2$$

and use the partitions of unity.

N 4-25-11

Remark N.1.

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\} = \overline{C_0^\infty(\Omega)}^{H^1}$$

Theorem N.2. ***Poincare Inequality***

$$\|u\|_{L^1(\Omega)} \leq c\|Du\|_{L^2(\Omega)} \qquad \forall\ u \in H_0^1(\Omega)$$

Corollary N.3.

There exist constants c_1, c_2 such that

$$c_1\|u\|_{H^1(\Omega)} \leq \|Du\|_{L^2(\Omega)} \leq c_2\|u\|_{H^1(\Omega)} \qquad \forall\ u \in H_0^1(\Omega)$$

$$\|u\|_{H_0^1(\Omega)} = \|Du\|^2_{L^2(\Omega)}$$

Definition N.4. *$\to$ in $H_0^1(\Omega)$*

$u_n \to u$ in $H_0^1(\Omega)$ iff $\|Du_n - Du\|_{L^2(\Omega)} \to 0$.

Definition N.5. *$\rightharpoonup$ in $H^1(\Omega)$*

$u_n \rightharpoonup u$ in $H^1(\Omega)$ iff $\langle u_n, \phi\rangle \rightharpoonup \langle u, \phi\rangle\ \forall\ \phi \in [H^1(\Omega)]'$

Remark N.6.

FACT:

$$[H^1(\mathbb{S}^1)]' = H^{-1}(\mathbb{S}^1)$$

Definition N.7. $H^{-1}(\Omega)$

$$H^{-1}(\Omega) = [H_0^1(\Omega)]'$$

Example N.8.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{N.1}$$

Definition N.9. ***Weak Solution***

u is a *weak solution* to (N.1) if

$$\int_\Omega Du \cdot Dv \, dx = \int_\Omega fv \, dx \qquad \forall \ v \in H_0^1(\Omega)$$

Equivalently,

$$(Du, Dv)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \tag{N.2}$$

Remark N.10.

For any $f \in L^2(\Omega)$ we have a unique solution to (N.1) because

$$(u, v)_{H_0^1(\Omega)} = \langle f, v \rangle_{H_0^1, H^{-1}} \qquad f \in H^{-1}(\Omega)$$

There exists a unique $u \in H_0^1(\Omega)$ solving (N.2) by the Riesz Representation Theorem.

Example N.11.

$$\begin{aligned} -\text{div}\big(A(x)Du\big) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ \frac{\partial}{\partial x_j}\left(A^{ij}(x)\frac{\partial u}{\partial x_j}\right) &= 0 \quad \text{in } \Omega \end{aligned} \tag{N.3}$$

NOTE: in previous example(s) we had $A^{ij} = [\text{Id}]^{ij}$, and thus $\int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx$

Definition N.12. $H_0^1(\Omega)$ ***Weak Solution***

u is an $H_0^1(\Omega)$ *weak solution* to (N.3) if

$$\int_\Omega A^{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx = \int_\Omega fv\,dx \left(\text{or } \langle f,v\rangle_{H_0^1,H^{-1}}\right) \qquad \forall v\in H_0^1(\Omega)$$

Remark N.13.

Suppose there exists $\lambda,\Lambda>0$ such that $\lambda\le A^{ij}(x)\le\Lambda$. We have an H^1-norm because

$$\lambda(Du,Dv)_{L^2(\Omega)} \le \underbrace{\int_\Omega A^{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx}_{H^1(\Omega)\text{ equivalent norm }\forall\ u\in H_0^1(\Omega)} \le \Lambda(Du,Dv)_{L^2(\Omega)}$$

Example N.14.

Let $\Omega=(0,1)$, $a(y)=$ 1-periodic function, $0<\lambda\le a(y)\le\Lambda$, $a^\epsilon(x)=a\left(\frac{x}{\epsilon}\right)$. Given $f\in L^2(0,1)$,

$$-\frac{d}{dx}\left(a^\epsilon(x)\frac{du^\epsilon}{dx}\right)=f \quad \text{in } (0,1)$$
$$u^\epsilon=0 \quad on\partial(0,1)\Rightarrow u^\epsilon(0)=u^\epsilon(1)=0$$

GOAL: $u^\epsilon\to u$ as $\epsilon\to 0$.
$a^\epsilon \overset{*}{\rightharpoonup} \overline{a}$ in $L^\infty(0,1)$, $\overline{a}=\int_0^1 a(y)\,dy$
GUESS: $-\frac{d}{dx}\left(\overline{a}\frac{du}{dx}\right)=-\overline{a}\frac{d^2u}{dx^2}=f \;\Rightarrow\;$ COMPLETELY WRONG!
ANSWER: $-\frac{1}{\overline{a^{-1}}}\frac{d^2u}{dx^2}=f$
In general: $\frac{1}{\int\frac{1}{a}\,dx}\le\int a$

Remark N.15.

Weak form: Given $f \in L^2(0,1)$, find $u \in H_0^1(0,1)$ such that

$$\int_0^1 a^\epsilon(x) \frac{du}{dx}\frac{dv}{dx}\, dx = \int_0^1 fv\, dx \qquad \forall\ v \in H_0^1(0,1)$$

1. $\forall\ \epsilon > 0$, there exists a unique solution $u^\epsilon \in H_0^1(\Omega)$
2. Let $v = u^\epsilon$e

$$\lambda \left\| \frac{du^\epsilon}{dx} \right\|_{L^2}^2 \leq \int_0^1 a^\epsilon(x) \frac{du^\epsilon}{dx}\frac{du^\epsilon}{dx}\, dx \leq \|f\|_{L^2}\|u^\epsilon\|_{L^2}$$

$$\lambda \|u^\epsilon\|_{H_0^1(0,1)}^2 \leq \|f\|_{L^2}\|u^\epsilon\|_{H_0^1(0,1)}$$

$$\|u^\epsilon\|_{H_0^1(0,1)} \leq \frac{1}{\lambda}\|f\|_{L^2}$$

$\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in H_0^1, so there exists a subsequence such that $u^{\epsilon'} \rightharpoonup u$ in $H_0^1(0,1)$.

Definition N.16. ***Def 1***

$$\langle u^\epsilon, \varphi \rangle_{H_0^1, H_{-1}} \to \langle u, \varphi \rangle_{H_0^1, H^{-1}}$$

Definition N.17. ***Def 2***

$$(u^\epsilon, v)_{H_0^1(0,1)} \to (u, v)_{H_0^1(0,1)} \qquad \forall\ v \in H_0^1(0,1)$$

(This is equivalent to Definition N.16 by the Riesz Representation Theorem)

Definition N.18. ***Def 3***

$u^\epsilon \rightharpoonup u$ in $H_0^1(0,1)$ iff

$$\int_0^1 \frac{du^\epsilon}{dx}\frac{dv}{dx}\, dx \to \int_0^1 \frac{du}{dx}\frac{dv}{dx}\, dx$$

Definition N.19. ***Def 4***

$u_n \rightharpoonup u$ in $H_0^1(\Omega)$ iff $Du_n \to Du$ in $L^2(\Omega)$.

Remark N.20.

The weak limit of a product is <u>not</u> the product of the weak limits.

Remark N.21.

$$\text{Let } \xi^\epsilon = a^\epsilon \frac{du^\epsilon}{dx}$$
$$-\frac{d}{dx}\xi^\epsilon = f \text{ in } L^2(0,1)$$
$$\xi^\epsilon \text{ is uniformly bounded in } H_1(0,1)$$
$$\xi^\epsilon \rightharpoonup \xi \text{ in } H^1(0,1)$$

Rellich's Theorem:

$$H^1(0,1) \hookrightarrow L^2(0,1) \text{ is compact}$$
$$\xi^\epsilon \to \xi \text{ in } L^2(0,1)$$

O 4-27-11

Example O.1. ***Weak Formulation, Variational Formulation***

$a^\epsilon(x) = a\left(\frac{x}{\epsilon}\right)$ and $a(y)$ is 1-periodic, $0 < \lambda \le a \le \Lambda$.
a^ϵ is uniformly bounded in $L^\infty(0,1)$.
$a^\epsilon \overset{*}{\rightharpoonup} \overline{a} = \int_0^1 a(y)\,dy$.
Sequence of solutions to

$$-\frac{d}{dx}\left(a^\epsilon(x)\frac{du^\epsilon}{dx}\right) = f \text{ in } (0,1) \tag{O.1}$$
$$u^\epsilon(0) = u^\epsilon(1) = 0$$

The obvious guess (see Example N.14) is wrong.

Step 0: (O.1) has a *weak formulation* or *variational formulation*

$$\int_0^1 a^\epsilon(x)\frac{du^\epsilon}{dx}\frac{dv}{dx}\,dx = \int_0^1 fv\,dx \qquad \forall\ v \in H_0^1(0,1)$$

Step 1: Let $v = u^\epsilon$. Then

$$\|u^\epsilon\|_{H_0^1(0,1)} \le \frac{1}{\lambda}\|f\|_{L^2(0,1)}$$

Then $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $H^1(0,1)$. By weak compactness, there exists a subsequence $u^\epsilon \rightharpoonup u$ in H_0^1:

$$\int_0^1 \frac{du^\epsilon}{dx}\phi\,dx \to \int_0^1 \frac{du}{dx}\phi\,dx \qquad \forall\ \phi \in L^2(0,1)$$

Step 2: Let $\xi^\epsilon = a^\epsilon \frac{du^\epsilon}{dx}$. This is uniformly bounded in $L^2(0,1)$ by the boundedness of $\{u^\epsilon\}_{\epsilon>0}$. Then

$$-\frac{d}{dx}\xi^\epsilon = f \text{ is uniformly bounded in } L^2(0,1) \tag{O.2}$$

Thus, ξ^ϵ is uniformly bounded in $H^1(0,1)$. Weak compactness implies that there exists a subsequence (same index used) $\xi^\epsilon \rightharpoonup \xi$ in $H^1(0,1)$.

Example O.2. ***Weak Formulation, Variational Formulation (Continued)***

Rellich's Strong Compactness: There exists a subsequence $\xi^\epsilon \to \xi$ in $L^2(0,1)$.

Notice that $\frac{du^\epsilon}{dx} = \frac{1}{a^\epsilon} \cdot \xi^\epsilon$. We know that $\frac{du^\epsilon}{dx} \rightharpoonup \frac{du}{dx}$ in $L^2(0,1)$.

$$\frac{1}{a^\epsilon} \cdot \xi^\epsilon \rightharpoonup a^{-1}\xi \tag{O.3}$$

We also know that $\frac{1}{a^\epsilon} \overset{*}{\rightharpoonup} \overline{a^{-1}}$ in $L^\infty(0,1)$ and $\xi^\epsilon \to \xi$ in $L^2(0,1)$.

$$\begin{aligned} \frac{du}{dx} &= \overline{a^{-1}}\xi \quad \Rightarrow \quad \xi = \frac{1}{\overline{a^{-1}}} \cdot \frac{du}{dx} \\ -\frac{d\xi}{dx} &= f \qquad \text{(from O.2)} \\ -\frac{d}{dx}\left(\frac{1}{\overline{a^{-1}}} \cdot \frac{du}{dx}\right) &= f \\ -\frac{1}{\overline{a^{-1}}} \cdot \frac{d^2u}{dx^2} &= f \end{aligned}$$

Proof. (Proof of O.3)
Goal: $\forall \phi \in L^2(0,1)$, $\int_0^1 \frac{1}{a^\epsilon}\xi^\epsilon \phi \, dx \to \int_0^1 \overline{a^{-1}}\xi\phi \, dx$, i.e.

$$\left| \int_0^1 \frac{1}{a^\epsilon}\xi^\epsilon \phi - \overline{a^{-1}}\xi\phi \, dx \right| \to 0 \quad \text{as } \epsilon \to 0$$

We compute:

$$\begin{aligned} \left| \int_0^1 \frac{1}{a^\epsilon}\xi^\epsilon \phi - \overline{a^{-1}}\xi\phi \, dx \right| &= \left| \int_0^1 (\xi^\epsilon - \xi)\frac{1}{a^\epsilon}\phi + \xi\left(\frac{1}{a^\epsilon} - \overline{a^{-1}}\right)\phi \, dx \right| \\ &\le \underbrace{\int_0^1 |\xi^\epsilon - \xi| \left|\frac{1}{a^\epsilon}\right| |\phi| \, dx}_{\text{I}} + \underbrace{\left| \int_0^1 \left(\frac{1}{a^\epsilon} - \overline{a^{-1}}\right)\xi\phi \, dx \right|}_{\text{II}} \end{aligned}$$

$$\text{I} \le \|\xi^\epsilon - \xi\|_{L^2} \left\| \frac{\phi}{a^\epsilon} \right\|_{L^2} \to 0 \tag{O.4}$$

where (O.4) is due to strong convergence of $\xi^\epsilon \to \xi$ in L^2 and the uniform L^∞ bound on $\frac{1}{a^\epsilon}$.

For II, we see that

$$\int_0^1 |\xi\phi| \, dx \le \|\xi\|_{L^2}\|\phi\|_{L^2} \quad \Rightarrow \quad \xi\phi \in L^1$$

$$\frac{1}{a^\epsilon} \overset{*}{\rightharpoonup} \overline{a^{-1}} \quad \text{in } L^\infty(0,1)$$

Thus, II $\to 0$. ∎

Theorem O.3. ***Rellich's Theorem (Strong Compactness, Arzela-Ascoli for $W^{1,p}$ Spaces)***

Given: $\Omega \subset \mathbb{R}^n$ bounded, smooth; $p < n$; $1 \leq q < \frac{np}{n-p}$.
For a uniformly bounded sequence $(u_j) \subset W^{1,p}(\Omega)$, there exists a subsequence $(u_{j_k}) \to u$ in $L^q(\Omega)$.
That is,

$$H^s(0,1) \hookrightarrow L^2(0,1) \qquad r < s$$

In the previous example, we used $H^1(0,1) \hookrightarrow L^2(0,1)$ compactly.

The proof of this theorem relies on Gagliardo-Nirenberg on bounded domains and Sobolev extension operators.

Theorem O.4. ***Sobolev Extension Theorem***

Let Ω be bounded and smooth, and let $\tilde{\Omega}$ also be bounded such that $\Omega \subset\subset \tilde{\Omega}$. There exists a continuous linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ with the following properties:

1. $Eu = u$ a.e. in Ω
2. spt $(Eu) \subset \tilde{\Omega}$
3. $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C(p, \Omega, \tilde{\Omega})\|u\|_{W^{1,p}(\Omega)} \qquad \forall\ u \in W^{1,p}$

P 4-29-11

Theorem P.1. ***Extension Theorem***

$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that for some $\tilde{\Omega},\ \Omega \subset\subset \tilde{\Omega}$,

1. $Eu = u$ a.e. in Ω
2. spt $(Eu) \subset \tilde{\Omega}$
3. $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C(p, \Omega, \tilde{\Omega})\|u\|_{W^{1,p}(\Omega)}$

Theorem P.2. ***Gagliardo-Nirenberg on Bounded Domains***

$1 \le p < n,\ p^* = \frac{np}{n-p}$

$$\|u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \qquad \forall\ u \in W^{1,p}(\Omega)$$

Proof. By the extension theorem,

$$\begin{aligned}\|u\|_{L^{p^*}(\Omega)} &\le \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \\ &\overset{\text{G.N.}}{\le} C\|D(Eu)\|_{L^p(\mathbb{R}^n)} \\ &\le C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \\ &\overset{\text{continuity}}{\le} C\|u\|_{W^{1,p}(\Omega)}\end{aligned}$$

□

Theorem P.3.

$W_0^{1,p}(\Omega),\ 1 \le q \le p^*$

$$\|u\|_{L^q(\Omega)} \overset{\text{Hölder}}{\le} \|u\|_{L^{p^*}(\Omega)} \le C\|Du\|_{L^p(\Omega)}$$

$C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, so we use a sequence $(u_j) \subset C_0^\infty(\Omega)$, extend by zero to $\mathbb{R}^n$, and use continuity of norms.

Theorem P.4.

$1 \le q < \infty$

$$\|u\|_{L^q(\Omega)} \le C(q)\|Du\|_{L^n(\Omega)} \qquad \forall\ u \in W^{1,n}(\Omega)$$

with $C(q) \to \infty$ as $q \to \infty$.

Theorem P.5.

$p > n$

$$\|U\|_{C^{0,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)} \qquad \gamma = 1 - \frac{n}{p}$$

Theorem P.6. ***Rellich's Theorem (Strong Compactness)***

$1 \leq p < n$, Ω bounded

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{is compact,} \qquad 1 \leq q < \frac{np}{n-p} = p^*$$

Proof. **Step 0:** $1 \leq r \leq s \leq t \leq \infty \;\Rightarrow$

$$\|u\|_{L^s(\Omega)} \leq \|u\|^{\alpha}_{L^p(\Omega)} \|u\|^{1-\alpha}_{L^t(\Omega)} \qquad \alpha \in [0,1] \text{ (Hölder)}$$

Goal:

$$\|W\|_{L^q(\Omega)} \leq \underbrace{\|W\|^{\alpha}_{L^1(\Omega)}}_{\substack{\text{small by properties} \\ \text{of convolution}}} \underbrace{\|W\|^{1-\alpha}_{L^{p^*}(\Omega)}}_{\text{G.N.}}$$

Given: $\sup \|u_j\|_{W^{1,p}(\Omega)} \leq M$
Want: $u_{j_n} \to u$ in $L^q(\Omega)$ **Know:** (Arzela-Ascoli) if $(u_j) \subset C^0(\overline{\Omega})$ is uniformly bounded and equicontinuous, then there exists $u_{j_k} \to u$

Pick an element $u_j \in W^{1,p}(\Omega)$. Extend it: $Eu_j \in C_0^{(}\tilde{\Omega})$, $\in C_0^{\infty}(\mathbb{R}^n)$, $Eu_j = u_j$ a.e. in Ω, $\eta_\epsilon * Eu_j \to Eu_j$ in $W^{1,p}(\Omega)$ as $\epsilon \to 0 \;\Rightarrow\; Eu_j = Eu$ a.e.

Step 1: $u_j \xrightarrow{\text{extend}} Eu_j = \overline{u}_j$
Step 2: Mollify

$$\overline{u}_j^{\epsilon} = \eta_\epsilon * \overline{u}_j \in C_0^{\infty}(\tilde{\Omega})$$

For fixed $\epsilon > 0$, $(\overline{u}_j^{\epsilon})$ is a) uniformly bounded and b)equicontinuous. (Hint: Young's Inequality)

$\overline{u}_j^{\epsilon} - \overline{u}_j$ is small in certain norms. $\|\overline{u}_j^{\epsilon} - \overline{u}_j\|_{L^q(\Omega)}$ is ridiculously small:

$$\begin{aligned}
\|\overline{u}_j^{\epsilon} - \overline{u}_j\|_{L^q(\Omega)} &\leq \|\overline{u}_j^{\epsilon} - \overline{u}_j\|^{\alpha}_{L^1(\tilde{\Omega})} \|\overline{u}_j^{\epsilon} - \overline{u}_j\|^{1-\alpha}_{L^{p^*}(\tilde{\Omega})} \\
&\overset{\text{G.N.}}{\leq} \|\overline{u}_j^{\epsilon} - \overline{u}_j\|^{\alpha}_{L^1(\tilde{\Omega})} \|D\overline{u}_j^{\epsilon} - D\overline{u}_j\|_{L^{p^*}(\tilde{\Omega})} \\
&\leq \|\overline{u}_j^{\epsilon} - \overline{u}_j\|_{L^1(\tilde{\Omega})} \cdot CM
\end{aligned}$$

$$|\overline{u}_j^{\epsilon}(x) - \overline{u}_j(x)| \leq \int_{B(0,\epsilon)} |\eta_\epsilon(y)||\overline{u}_j(x-y) - \overline{u}_j(x)|\, dy \tag{P.1}$$

Recall that

$$\eta_\epsilon(y) = \frac{1}{\epsilon^n}\left(\frac{y}{\epsilon}\right) \quad \Rightarrow \quad z = \frac{y}{\epsilon} \quad \Rightarrow \quad dy = \eta^n dz$$

Thus, continuing from (O.1), we have

$$\int_{B(0,\epsilon)} |\eta_\epsilon(y)||\overline{u}_j(x-y) - \overline{u}_j(x)|\,dy = \int_\Omega \int_{B(0,1)} |\eta(z)|\overline{u}_j(x-\epsilon z) - \overline{u}(x)|\,dz\,dx$$
$$= \int_{B(0,1)} \eta(z) \left| \int_0^1 \frac{d}{dt}\overline{u}_j(x-\epsilon tz)\,dt \right| dz\,dx \leq \epsilon C$$

We get that $(\overline{u}_j^\epsilon)$ is uniformly bounded by Young's Inequality:

$$r = \infty \qquad \|\eta_\epsilon\|_{L^q} < \infty$$
$$\|\overline{u}_j^\epsilon\|_{L^\infty} \leq \|\eta_\epsilon\|_{L^\infty} \underbrace{\|\overline{u}_j\|_{L^1}}_{\text{Hölder}} \sim \frac{C}{\epsilon^n} \qquad \text{(uniform in } j)$$
$$\|D\overline{u}_j^\epsilon\|_{L^\infty} \leq \frac{C}{\epsilon^{n+1}} \qquad \text{(uniform in } j)$$
$$\|\overline{u}_{j_k}^\epsilon - \overline{u}_{j_l}^\epsilon\|_{L^q(\tilde{\Omega})} \leq C\epsilon$$

Let $\epsilon = \frac{1}{n}$ and use a diagonal argument. □

Problem P.7. ***10-15 min. (3 such problems on Midterm)***

$u_j \rightharpoonup u$ in $W_0^{1,1}(0,1)$
Show $u_j \to u$ a.e.
$u_j \rightharpoonup u$ weakly in $W_0^{1,1}(0,1)$ if

$$\frac{du_j}{dx} \rightharpoonup \frac{du}{dx} \text{ in } L^1(0,1)$$

Remark P.8. ***Midterm Comment***

Shkoller is tempted to give a problem on computing a weak derivative, but he probably won't. *BUT* you should know how to compute

$$\frac{\partial}{\partial x_i}|x|$$

Q 5-6-11

Remark Q.1. ***Test Question 1***

Morrey's inequality:

$$\begin{aligned} |u(x)-u(y)| &\le C r^{1-n/p}\|Du\|_{L^p} && \forall\ y\in B(x,r) \\ \|u^\epsilon-u\|_{L^\infty} &\le C\epsilon^{1-n/p}\|Du\|_{L^p} && n=3,\ p=6\Rightarrow \sqrt{\epsilon} \\ &\le C\sqrt{\epsilon}\|Du\|_{L^6(\mathbb{R}^3)} \\ &\overset{\text{G.N.}}{\le} C\sqrt{\epsilon}\|D^2u\|_{L^2(\mathbb{R}^3)} \\ &\overset{\text{def}}{\le} C\sqrt{\epsilon}\|u\|_{H^2(\mathbb{R}^3)} \end{aligned}$$

Remark Q.2. ***Test Question 2***

$$G(x)=-\frac{1}{2\pi}\log|x| \qquad (\Delta G=\delta)$$

$$\begin{aligned} \text{Show:}\qquad f(x) &= \lim_{\epsilon\to0}\Bigg(\underbrace{\int_{B(0,\epsilon)} G(y)\Delta_y f(x-y)\,dy}_{\text{I}} + \underbrace{\int_{\mathbb{R}^2-B(0,\epsilon)} G(y)\Delta_y f(x-y)\,dy}_{\text{II}}\Bigg) \\ &= \lim_{\epsilon\to0}\int_{R^2} G(y)\Delta_y f(x-y)\,dy \\ &= G*f \end{aligned}$$

$$f(x)=\lim_{r\to0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy = \lim_{r\to0}\frac{1}{|\partial B(x,r)|}\int_{|\partial B(x,r)|} f(y)\,dS(y)$$

$$\text{I}=\lim_{\epsilon\to0}\int_0^{2\pi}\int_0^{\epsilon}\log r\Delta_y f(x-y) r\,dr\,d\theta \xrightarrow{\text{DCT}} 0$$

$$\frac{\partial G}{\partial x_i} = -\frac{1}{2\pi}\frac{1}{|x|}\frac{x_i}{|x|} = -\frac{x_i}{2\pi}\frac{1}{|x|^2}$$

$$\begin{aligned} \text{II} &= \int_{\mathbb{R}^2-B(0,\epsilon)}\frac{1}{2\pi}\frac{y_i}{|y|^2}\frac{\partial}{\partial y_i}f(x-y)\,dy - \frac{1}{2\pi}\int_{\partial B(0,\epsilon)}\underbrace{\frac{y_i}{|y|^2}N_i}_{\frac{1}{\epsilon}}\frac{\partial f}{\partial y_i}(x-y)\,\underbrace{dS(y)}_{\epsilon d\theta} \\ &= \frac{1}{2\pi}\int_{\partial B(0,\epsilon)}\frac{y_i}{|y|^2}\frac{y_i}{|y|}f(x-y)\,dS(y) \\ &= \frac{1}{2\pi\epsilon}\int_{\partial B(0,\epsilon)} f(x-y)\,dS(y) \end{aligned}$$

Q.1 Fourier Transform

Definition Q.3. ***Fourier Transform, $\mathcal{F}$***

For $u \in L^2(\mathbb{R}^n)$, we define

$$\mathcal{F}u(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-ix\xi}\, dx$$

Note: $\mathcal{F}u \in L^\infty(\mathbb{R}^n)$ by Hölder's inequality.

Remark Q.4. ***Fourier Transform, $L^2(\mathbb{R}^n)$ Case***

$\mathcal{F} : L^2 \to L^2$ is an isometric isomorphism

Question: why does $\mathcal{F}$ make sense on $L^2(\mathbb{R}^n)$?

Given $u \in L^2(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} |u|^2\, dx < \infty \not\Rightarrow \int_{\mathbb{R}^n} |u|\, dx < \infty$$

Answer: the Gaussian, $g(x) = ce^{-|x|^2}$.
To make sense of this, we introduce the *Tempered Distribution*:

$$\begin{aligned} S(\mathbb{R}^n) &= \left\{ u \in C^\infty(\mathbb{R}^n) \;\middle|\; x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n) \ \forall\ \alpha, \beta \in \mathbb{Z}^n_+ \right\} \\ &= \text{the functions of rapid decay} \\ S'(\mathbb{R}^n) &= \text{dual space} = \text{tempered distributions} \\ \mathcal{F} &: S'(\mathbb{R}^n) \to S'(\mathbb{R}^n) \end{aligned}$$

On $S(\mathbb{R}^n)$, $\mathcal{F} \circ \mathcal{F}^* = \mathrm{Id} = \mathcal{F}^* \circ \mathcal{F}$.

Definition Q.5. ***Inverse Fourier Transform***

$$\mathcal{F}^* u(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{ix/xi}\, dx$$

R 5-9-11

Definition R.1.

$f \in L^1(\mathbb{R}^n)$

$$\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y)e^{-iy\xi}\, dy$$
$$\mathcal{F}^* f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\xi)e^{ix\xi}\, dx$$

Definition R.2.

$S(\mathbb{R}^n) = \text{rapidly decaying} = \{u \in C^\infty(\mathbb{R}^n) \bigm| x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n),\ \alpha, \beta \in \mathbb{Z}_+^n\}$

Remark R.3.

FACT: $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$

$|\xi^\beta D_\xi^\alpha \mathcal{F}f(\xi)| = |\mathcal{F}(D^\beta x^\alpha f)|$

Remark R.4. ***Notation***

$\hat{f}(\xi) = \mathcal{F}f(\xi)$

Example R.5.

$$\begin{aligned}\frac{\partial}{\partial \xi_j} &= (2\pi)^{-n/2} \frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^n} e^{-iy\xi} f(y)\, dy \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} -iy_j e^{-iy\xi} f(y)\, dy \\ &= \mathcal{F}(-iy_j f)y))\end{aligned}$$

Example R.6.

$$\begin{aligned}\xi_j \hat{f}(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \xi_j e^{-iy\xi} f(y)\, dy \\ &= (2\pi)^{-n/2} i \int_{\mathbb{R}^n} \frac{\partial}{\partial y_j} e^{-iy\xi} f(y)\, dy \\ &= -i2\pi^{-n/2} \int_{\mathbb{R}^n} e^{-iy\xi} \frac{\partial f}{\partial y_j}(y)\, dy\end{aligned}$$

No boundary terms since $f \in S(\mathbb{R}^n)$.

Remark R.7.

FACT: $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$

Example:

$$G(x) = (2\pi)^{-n/2} e^{-|x|^2/2} \in S(\mathbb{R}^n)$$

Since $\mathcal{D} \subset S,\ S' \subset \mathcal{D}'$.

Lemma R.8.

For $u, v \in S(\mathbb{R}^n)$, we have that

$$(\mathcal{F}u, v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{F}^* v)_{L^2(\mathbb{R}^n)}$$

Remark R.9.

FACT: $\mathcal{F}^*$ is the L^2 adjoint of $\mathcal{F}$.

Theorem R.10.

$$\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = \mathrm{Id} \quad \text{on } S(\mathbb{R}^n)$$

Remark R.11.

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

Proof. **Want to prove:**

$$\mathcal{F}^*\mathcal{F}f(x) = f(x) \quad \forall\ f \in S(\mathbb{R}^n)$$

$$\begin{aligned}\mathcal{F}^*\mathcal{F}f(x) &= 2\pi^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)e^{-iy\xi}\,dy e^{ix\xi}\,d\xi\\ &= 2\pi^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\xi(x-y)}f(y)\,dy\,d\xi\\ &\overset{\text{DCT}}{=} \lim_{\epsilon\to 0} 2\pi^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2}e^{i\xi(x-y)}f(y)\,dy\,d\xi\\ &\overset{\text{Fubini}}{=} \lim_{\epsilon\to 0} 2\pi^{-n}\int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2+i\xi(x-y)}\,d\xi\,dy\end{aligned}$$

Let

$$K_\epsilon(x) = 2\pi^{-n}\int_{\mathbb{R}^n} e^{-\epsilon|xi|^2+ix\xi}\,d\xi$$

Then

$$\begin{aligned}\mathcal{F}^*\mathcal{F}f(x) &= \lim_{\epsilon\to 0} K_\epsilon * f\\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} K_\epsilon(x-y)f(y)\,dy\end{aligned}$$

Recall: standard mollifier

$$\begin{aligned}\rho_1(x)\ \text{spt}\ \rho_1 &\subset \overline{B(0,1)}\\ \rho_\delta(x) &= \frac{1}{\delta^n}\rho\left(\frac{x}{\delta}\right)\\ \int_{\mathbb{R}^n}\rho_\delta(x)\,dx &= 1\\ \delta &= \sqrt{\epsilon}\end{aligned}$$

$$\begin{aligned}K_1(x) &= 2\pi^{-n}\int_{\mathbb{R}^n} e^{-|\xi|^2+ix\xi}\,d\xi\\ K_{1/2}(x) &= 2\pi^{-n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}|\xi|^2}e^{ix\xi}\,d\xi\\ &= \mathcal{F}\left(2\pi^{-n/2}e^{-\frac{1}{2}|\xi|^2}\right)\end{aligned}$$

Claim:

$$K_{1/2}(x) = -\frac{1}{2}e^{-|x|^2/2} \equiv G(x) \tag{R.1}$$

In other words, the claim says that $G = \mathcal{F}G$.

Then in 1-D:

$$\frac{d}{dx}G(x) + xG(x) = 0$$

Keep in mind that

$$\begin{aligned} e^{-|x|^2/2} &= e^{-x_1^2/2 - x_2^2/2 - \cdots - x_n^2/2} \\ &= e^{-x_1^2/2} e^{-x_2^2/2} \cdots e^{-x_n^2/2} \end{aligned}$$

Compute the Fourier transform of (R.1):

$$-i\left(\frac{d}{d\xi}\hat{G}(\xi) + \xi\hat{G}(\xi)\right) = 0$$

Thus,

$$\hat{G}(\xi) = Ce^{-|\xi|^2/2}$$

Recap: We wrote it out, used an integrating factor via DCT, used Fubini to write it as convolution with kernel K, where $K_\epsilon = \frac{1}{(C\epsilon)^{n/2}} K\left(\frac{x}{\sqrt{\epsilon}}\right)$. And we get that $\mathcal{F}^*\mathcal{F}f = f$. $\square$

S 5-10-11 (Section)

Example S.1.

$$\begin{aligned} \Delta u &= 0 \qquad \text{on } \Omega \text{ bounded, open, connected} \\ u|_{\partial\Omega} &= f \qquad f \in C(\partial\Omega),\ \partial\Omega \text{ is } C^1 \end{aligned}$$

Prove that the solution is unique.

Let u_1, u_2 be solutions. Take $u = u_1 - u_2$. Then

$$\begin{aligned} \Delta u &= 0 \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

Remark:

$$\int_\Omega u\Delta v - Du \cdot Dv\, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial n}\, dS$$

$$\begin{aligned} \int_\Omega (\cancel{u\Delta u} - Du \cdot Du)\ dx &= \cancel{\int_{\partial\Omega} u \frac{\partial u}{\partial n}\, dS} \\ \int_\Omega |Du|^2\, dx &= 0 \\ |Du| &= 0 \qquad \text{on } \Omega \\ Du &= 0 \qquad \text{on } \Omega \end{aligned}$$

Thus, u is a locally constant function: $u = c$.

$x_0 \in \Omega$.

$$\Omega' = \{x \mid u(x) = u(x_0)\} \subseteq \Omega \qquad \Rightarrow \qquad \Omega' = \Omega$$

Ω' is closed. Ω' is open (Prove!).

Example S.2. $f' = 0$

$$\Omega = (0,1) \cup (3,4)$$

$$f = \begin{cases} c_1 & \text{on } (0,1) \\ c_2 & \text{on } (3,4) \end{cases}$$

Lemma S.3.

Let f be a nice (smooth, C^∞) function.

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

Then

$$\widehat{f * g} = \hat{f} * \hat{g}$$

Proof.

$$\begin{aligned}
\widehat{f * g} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f * g e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-iky} g(x-y)\, dy \right) e^{-ik(x-y)}\, dx \\
&\overset{\text{Fubini}}{=} \hat{f} * \hat{g}
\end{aligned}$$

□

Remark S.4. ***Solving the Heat Equation with the Fourier Transform***

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$u(x,0) = f(x)$$

$$-|k|^2 \hat{u}(k,y) + \frac{d^2}{dy^2} u(k,y) = 0$$
$$\hat{u}(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-ikx}\, dx$$
$$\hat{u}(k,y) = \underbrace{\cancel{c_1(k) e^{|k|y}}}_{\substack{\text{Riemann-Lebesgue} \\ \text{Lemma}}} + c_2 e^{-|k|y}$$
$$= \hat{f}(k) e^{-|k|y}$$
$$u(x,y) = P_y * f$$
$$\hat{P}_y = e^{-|k|y}$$

Calculate the inverse Fourier transform of $\hat{P}_y$.

$$P_y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|k|y} e^{ikx}\, dk$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|k|y} \left(\cos kx + \underbrace{\cancel{i \sin kx}}_{\text{even/odd}} \right) dk$$
$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ky} \cos kx\, dk$$
$$= \frac{2}{\sqrt{2\pi}} \frac{1}{x^2+y^2} e^{-ky} \left(k \sin x - y \cos kx \right) \Big|_{k=0}^{\infty}$$
$$= \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2}$$

Plugging back in to our equation for $u = P_y * f$, we get

$$u(x,y) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} f(y)\, dy$$

Remark S.5. ***Proving the Fourier Inverse Transform***

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}(t)e^{ixt}e^{-\epsilon^2t^2}\,dt = \phi_\epsilon(x)$$
$$= \phi * \eta_\epsilon(x) \xrightarrow{\text{uniformly}} \phi \in S(\mathbb{R})$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}(t)e^{ixt}\,dt$$

We know:

- $\widehat{e^{-kx^2}} =$ Gaussian
- $\widehat{f * g} = \hat{f}\hat{g}$

T 5-11-11

Theorem T.1. ***(From Last Time)***

$$\mathcal{F}^*\mathcal{F} = \mathrm{Id} = \mathcal{F}\mathcal{F}^* \qquad \text{on } S(\mathbb{R}^n)$$

Consequence:

$$(\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{F}^*\mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)} \quad \forall\, u, v \in S(\mathbb{R}^n)$$
$$\|\mathcal{F}u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \tag{T.1}$$

Definition T.2.

Let $(u_j) \subset S(\mathbb{R}^n)$ such that $u_j \to u$ in $L^2(\mathbb{R}^n)$.

$$\mathcal{F}u = \lim_{j\to\infty} \mathcal{F}u_j \qquad \text{for } u \in L^2(\mathbb{R}^n)$$

This is independent of the approximating sequence that you take. This is because of (T.1).

Corollary T.3.

$$\|\mathcal{F}u\|_{L^2} = \|u\|_{L^2} \quad \forall\, u \in L^2(\mathbb{R}^n) \quad \xrightarrow{\text{polarization}} \quad (\mathcal{F}u, \mathcal{F}v)_{L^2} = (u, v)_{L^2}$$

Example T.4.

$$x \mapsto e^{-t|x|}, \quad t > 0,\ x \in \mathbb{R}^n$$

Does this have rapid decay? Yes.

Remark T.5. ***Topology of*** $S(\mathbb{R}^n)$

$S(\mathbb{R}^n)$ is a Frechet space with semi-norm

$$p_k(u) = \sup_{x\in\mathbb{R}^n,\ |\alpha|\le k} \sqrt{1+|x|^2}^{\,k} |D^\alpha u(x)|$$

and distance function

$$d(u, v) = \sum_{k=0}^{\infty} 2^{-k} \frac{p_k(u-v)}{1+p_k(u-v)}$$

Definition T.6. ***Convergence in $S(\mathbb{R}^n)$***

$u_j \to u$ in $S(\mathbb{R}^n)$ if $p_k(u_j - u) \to 0$ as $j \to \infty \ \forall \ k \geq 0$.

Definition T.7. ***Continuous Linear Functional on $S(\mathbb{R}^n)$, Tempered Distribution***

$T : S(\mathbb{R}^n) \to \mathbb{R}$,

$$|\langle T, u \rangle| \leq C p_k(u) \qquad \text{for some } k \geq 0$$

$S'(\mathbb{R}^n) =$ dual space of $S(\mathbb{R}^n) =$ *tempered distributions*

Definition T.8.

$\mathcal{F} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$

$$\langle \mathcal{F}T, u \rangle = \langle T, \mathcal{F}u \rangle$$

Example T.9.

$\delta \in S'(\mathbb{R}^n)$, where $\langle \delta, u \rangle = u(0)$, $\langle \delta_x, u \rangle = u(x)$

$$\langle \mathcal{F}\delta, u \rangle = \langle \delta, \mathcal{F}u \rangle = \mathcal{F}u(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i \cdot 0 \cdot x} u(x)\, dx$$

Remark T.10.

$i : L^p(\mathbb{R}^n) \to S'(\mathbb{R}^n)$, and

$$\langle f, u \rangle = \int_{\mathbb{R}^n} f(x) u(x)\, dx$$

$$\mathcal{F}\delta = (2\pi)^{-n/2} \qquad \text{in } S'(\mathbb{R}^n)$$

Example T.11. ***Fourier Transform***

Compute the Fourier transform of $e^{-t|x|},\ t>0,\ x\in\mathbb{R}^n$.
$n=1$:

$$\begin{aligned}
\mathcal{F}(e^{-t|x|}) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t|x|}e^{-ix\xi}\,dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{tx}e^{-ix\xi}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-tx}e^{-ix\xi}\,dx\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{t-i\xi}e^{x(t-i\xi)}|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}}\frac{-1}{t+i\xi}e^{-x(t+i\xi)}|_{0}^{\infty}\\
&= \frac{1}{\sqrt{2\pi}}\frac{2t}{t^2+\xi^2}\\
&= \sqrt{\frac{2}{\pi}}\frac{t}{t^2+\xi^2}
\end{aligned}$$

$n>1$:
Guess:

$$e^{-t|x|} = \int_0^\infty g(t,s)e^{-s|x|^2}\,ds$$

Take the Fourier transform of this guess:

$$\mathcal{F}(e^{-t|x|}) = \int_0^\infty \mathcal{F}(e^{-s|x|^2})\,ds$$

We know that

$$\mathcal{F}((2\pi)^{-n/2}e^{-|x|^2/2}) = 2\pi^{-n/2}e^{-|x|^2/2}$$

Then

$$\mathcal{F}(e^{-s|x|^2}) = \underbrace{a_\pi\sqrt{\frac{1}{s}}^{\,n}e^{-|\xi|^2/4s}}_{\hat{u}(\xi)}$$

and we have

$$\begin{aligned}
\mathcal{F}(e^{-t|x|}) &= \int_0^\infty g(t,s)a_\pi\sqrt{\frac{1}{s}}^{\,n}e^{-|\xi|^2/4s}\,dx\\
\mathcal{F}(e^{-t\lambda}) &= \frac{1}{\pi}\int_{-\infty}^{\infty}e^{-t\lambda}e^{i\lambda\xi}\,d\xi \qquad \text{where } \lambda=|x|>0
\end{aligned}$$

Verify that

$$\begin{aligned}
\int_0^\infty e^{-s(t^2+\xi^2)}\,ds = -\frac{1}{t^2+\xi^2}e^{-s(t^2+\xi^2)}|_0^\infty &= \frac{1}{t^2+\xi^2}\\
&= \sqrt{\frac{2}{\pi}}t\int_0^\infty e^{-st^2}e^{-s\xi^2}\,ds
\end{aligned}$$

Then we have that

$$e^{-t\lambda} = \mathcal{F}^*\left(\frac{t}{t^2+\xi^2}\sqrt{\frac{2}{\pi}}\right)$$

U 5-13-11

Example U.1. $\mathcal{F}(e^{-t|x|})$

1-D:

$$\mathcal{F}(e^{-t|x|}) = \sqrt{\frac{2}{\sqrt{\pi}}}\frac{t}{t^2+\xi^2}, \qquad t>0$$

2-D:

$$\int_0^\infty e^{-st^2}e^{-s\xi^2}\,ds = \frac{1}{t^2+\xi^2}$$

Combining 1-D and 2-D:

$$\begin{aligned}
\mathcal{F}(e^{-t|x|}) &= \sqrt{\frac{2}{\pi}}t\int_0^\infty e^{-st^2}e^{-s\xi^2}\,ds \\
e^{-t|x|} &= \sqrt{\frac{2}{\pi}}t\int_0^\infty e^{-st^2}\mathcal{F}^*(e^{-s\xi^2})\,ds \\
\mathcal{F}^*(e^{-s\xi^2}) &= \qquad\qquad\qquad (U.1)
\end{aligned}$$

Use that

$$\begin{aligned}
\mathcal{F}\left(\frac{1}{\sqrt{2\pi}^n}e^{-|x|^2/2}\right) &= \frac{1}{\sqrt{2\pi}^n}e^{-|\xi|^2/2} \\
\frac{1}{\sqrt{2\pi}}e^{-x^2/2} &= \mathcal{F}^*\left(\frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}\right) \qquad (U.2)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}e^{ix\cdot\xi}\,d\xi
\end{aligned}$$

Goal:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-s\xi^2}e^{ix\xi}\,d\xi \quad\Rightarrow\quad \int_{-\infty}^\infty e^{-y^2/2}e^{ixy/\sqrt{2s}}\frac{1}{\sqrt{2s}}\,dy \qquad (U.3)$$

$$\begin{aligned}
-s\xi^2 &= -y^2/2 \\
y &= \sqrt{2s}\xi \\
\xi &= \frac{y}{\sqrt{2s}} \\
dy &= \sqrt{2s}\,d\xi \\
d\xi &= \frac{1}{\sqrt{2s}}\,dy \\
(U.3) &= \frac{e^{-\left(\frac{x}{\sqrt{2s}}\right)^2/2}}{\sqrt{2s}} \\
&= \frac{e^{-|x|^2/4s}}{\sqrt{2s}}
\end{aligned}$$

Example U.2. ***. . . Continued***

3:

$$\mathcal{F}\left(e^{-s|\xi|^2}\right) = \frac{1}{\sqrt{2s}^n} e^{-|x|^2/4s}$$

$$e^{-t|x|} = \int_0^\infty e^{-st^2} \frac{1}{\sqrt{2s}^n} e^{-|x|^2/4s}\, ds$$

Guess: $n \geq 1$

$$e^{-t|x|} = \int_0^\infty \frac{1}{\sqrt{2s}^n} g(t,s) e^{-|x|^2/4s}\, ds$$

Goal: find $g(t,s)$.

$$\begin{aligned}
\lambda &= |x| \geq 0 \\
e^{-t\lambda} &= \mathcal{F}^*\left(\sqrt{\frac{2}{\pi}} \frac{t}{t^2 + |\xi|^2}\right) \\
&= \frac{1}{\pi} \int_{-\infty}^\infty \frac{t}{t^2 + |\xi|^2} e^{i\lambda\xi}\, ds \\
&= \frac{1}{\pi} \int_{-\infty}^\infty t \int_0^\infty e^{-st^2} e^{-s\xi^2}\, ds e^{i\lambda\xi}\, d\xi \\
&= \frac{1}{\pi} \int_0^\infty te^{-st^2} \int_{-\infty}^\infty e^{-s\xi^2} e^{i\lambda\xi}\, d\xi\, ds \\
&= a_{\pi,n} \int_0^\infty t\sqrt{s}^{-n} e^{-|x|^2/4s}\, ds \\
\mathbf{a}_\pi \frac{1}{\sqrt{s}^n} g(t,s) &= te^{-st^2} \sqrt{s}^{-1} \\
g(t,s) &= a_{\pi,n} te^{-st^2} \sqrt{s}^{n-1}
\end{aligned}$$

Thus,

$$\mathcal{F}\left(e^{-t|x|}\right) = \int_0^\infty a_\pi t\sqrt{s}^{n-1} e^{-st^2} e^{-s\xi^2}\, ds$$

Remark U.3.

$$\begin{aligned}
\mathcal{F}(e^{-t|x|}) &= a_{\pi,n} \frac{t}{(t^2 + |\xi|^2)^{\frac{n+1}{2}}} \int_0^\infty s^{\frac{n-1}{2}} e^{-s}\, ds \\
&= \frac{a_{\pi,n} t}{(t^2 + |\xi|^2)^{\frac{n+1}{2}}} \gamma\left(\frac{n+1}{2}\right)
\end{aligned}$$

V 5-16-11

Remark V.1. ***Fundamental Solution to $-\Delta u = f$ in $\mathbb{R}^3$***

$$-\Delta u = \sum_{i=1}^{3} -\frac{\partial^2 u}{\partial x_i^2}$$

$$\mathcal{F}(-\Delta u) = \mathcal{F}(f) \quad \Leftrightarrow \quad |\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$$

$$\hat{u}(\xi) = \frac{1}{|\xi|^2} \hat{f}(\xi)$$

The solution is given by applying $\mathcal{F}^*$:

$$u(x) = \mathcal{F}^* \hat{u} = \mathcal{F}^* \left(\frac{1}{|\xi|^2} \hat{f}(\xi) \right)$$

$$u(x) = c\mathcal{F}^* \left(\frac{1}{|\xi|^2} \right) * f$$

$\mathcal{F}, \mathcal{F}^* \xrightarrow{\text{multiplication}}$ convolution, and the converse is also true.

$$\mathcal{F}^* \left(\frac{1}{|\xi|^2} \right) = -\frac{c}{4\pi} \frac{1}{|x|}$$

$$\text{(in 3-D)} \; = -\Delta u = f$$

$$u(x) = c \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) \, dy$$

Green's Function:

$$G(x) = \frac{c}{|x|}$$

Remark V.2. ***Last Time***

$$\gamma\left(\frac{n+1}{2}\right) = \int_0^\infty s^{\frac{n}{2}-\frac{1}{2}} e^{-s}\,ds$$

$$\gamma(\beta) = \int_0^\infty s^{\beta-1} e^{-s}\,ds$$

Let's look at the integral

$$\int_0^\infty s^{-1/2} e^{-s|x|^2}\,ds = |x|^{-1}\gamma\left(\frac{1}{2}\right)$$

$$t = s|x|^2, \quad s = t|x|^{-2}$$

$$ds = |x|^{-2}\,dt$$

$$|x|^{-1} = \frac{1}{\gamma\left(\frac{1}{2}\right)} \int_0^\infty s^{-1/2} e^{-s|x|^2}\,ds$$

$$\mathcal{F}(|x|^{-1}) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} \mathcal{F}(e^{-s|x|^2})\,ds$$

$$\mathcal{F}(|x|^{-1}) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} \frac{1}{\sqrt{2s}^3} e^{-|\xi|^2/4s}\,ds$$

$$= \frac{1}{\sqrt{\pi}\sqrt{2}^3} \int_0^\infty s^{-2} e^{-|\xi|^2/4s}\,ds$$

$$t = |\xi|^2/4s, \; s = t^{-1}\frac{|\xi|^2}{4}$$

$$ds = -t^{-2}\frac{|\xi|^2}{4}\,dt$$

$$= \frac{1}{\sqrt{\pi}\sqrt{2}^3} \int_0^\infty t|\xi|^{-4} e^{-t} t^2 |\xi|^2\,ds$$

$$= \gamma(1)\sqrt{\frac{2}{\pi}}|\xi|^{-2}$$

Thus,

$$\mathcal{F}(|x|^{-1}) = c|\xi|^{-2}, \quad u(x) = c\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\,dy$$

whenever $-\Delta u = f$ in $\mathbb{R}^3$.

$$-\Delta u = f \text{ in } S'(\mathbb{R}^3)$$

$$-\Delta\left(\frac{1}{|x|}\right) = c\delta \text{ in } S'(\mathbb{R}^3)$$

$$\hat{u}(\xi) = c\frac{\hat{f}}{|\xi|^2} + \delta$$

Not all solutions decay fast enough at $\pm\infty$. The Fourier transform in $L^2(\mathbb{R}^n)$ gives uniqueness.

Definition V.3. $\langle\ \rangle$

$$\langle \xi \rangle = \sqrt{1+|\xi|^2}$$

Using this notation, we have

$$H^k(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \langle \xi \rangle^k |\hat{u}(\xi)|^2 \, d\xi < \infty \right\}$$

Old:

$$H^2(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \left(|u(x)|^2 + |Du(x)|^2 \right) \, dx < \infty \right\}$$

New:

$$H^1(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} (1+|\xi|^2)|\hat{u}(\xi)|^2 \, d\xi < \infty \right\}$$

Example V.4. $\mathbb{R}^1$

$$H^1(\mathbb{R}^1) = \{ u \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} |\hat{u}(\xi)|^2 + \xi^2 |\hat{u}(\xi)|^2 \, d\xi < \infty \}$$
$$\int_{\mathbb{R}} \left(|u(x)|^2 + \left| \frac{du}{dx}(x) \right|^2 \right) \, dx < \infty \}$$

Example V.5. $\mathbb{R}^2$

$$\begin{aligned} H^1(\mathbb{R}^2) &= \left\{ u \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \left(|u(x)|^2 + \left| \frac{\partial u}{\partial x_1}(x) \right|^2 + \left| \frac{\partial u}{\partial x_2}(x) \right|^2 \right) \, dx < \infty \right\} \\ &= \left\{ u \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \left(|u(\xi)|^2 + |\xi_1|^2 |\hat{u}(\xi)|^2 + |\xi_2|^2 |\hat{u}(\xi)|^2 \right) \, d\xi < \infty \right\} \end{aligned}$$

Definition V.6. ***Functions with 1/2 derivative in*** $L^2(\mathbb{R}^n)$

$$H^{1/2}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \sqrt{1+|\xi|^2} |\hat{u}(\xi)|^2 \, d\xi < \infty \right\}$$

Theorem V.7. ***Trace Theorem***

Given: $u(\mathbf{x}) = u(x_1, x_2)$, define $f(x_2) = u(0, x_2)$.
Old: $T : H^1(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$
New: $T : H^1(\mathbb{R}^2) \to H^{1/2}(\mathbb{R})$ continuous, linear

General Trace Theorem: $s > 1/2,\ T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$ continuous, linear
Also, T is onto.

W 5-18-11

Theorem W.1. ***Trace Theorem***

$$T : H^1(\mathbb{R}^n) \to H^{1/2}(\mathbb{R}^{n-1}) \quad \text{continuously}$$

More generally:

$$T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1}) \quad \text{continuously for } s > \frac{1}{2}$$

Lemma W.2.

$u \in C(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), \qquad u = u(x_1, x_2), \qquad f(x_2) = u(0, x_2).$
Then for all $u \in C$ we have that

$$\hat{f}(\xi_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi_2)\, d\xi_2 \qquad \text{(average over } \xi_1\text{)}$$

Proof.

$$\begin{aligned}
\hat{f}(\xi_2) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x_2) e^{-ix_2\xi_2}\, dx_2 \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(0, x_2) e^{-ix_2\xi_2}\, dx_2 \\
u(x_1, x_2) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{u}(\xi_1, \xi_2) e^{ix_1\xi_1} e^{ix_2\xi_2}\, d\xi_1\, d\xi_2 \\
u(0, x_2) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{u}(\xi_1, \xi_2) e^{ix_2\xi_2}\, d\xi_1\, d\xi_2
\end{aligned}$$

□

Proof of Trace Theorem (W.1)

Proof. Want:

$$\|f\|_{H^{1/2}(\mathbb{R})} \leq C \|u\|_{H^1(\mathbb{R}^2)} \qquad \forall u \in H^1(\mathbb{R}^2)$$

Fourier:

$$\int_{\mathbb{R}} \sqrt{1+\xi_2^2} |\hat{f}(\xi_2)|^2 \, d\xi_2 \leq C \int_{\mathbb{R}^2} \langle \xi \rangle^2 |\hat{u}(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2$$

$$\hat{f}(\xi_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\xi_1}} \hat{u}(\xi_1, \xi_2) \, d\xi_1 \qquad = \int_{\mathbb{R}_{\xi_1}} \hat{u}(\xi_1, \xi_2) \langle \xi \rangle \langle \xi \rangle^{-1} \, d\xi_1$$

$$|\hat{f}(\xi_2)|^2 \leq \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\xi_1}} \int_{\mathbb{R}_{\xi_1}} |\hat{u}(\xi_1, \xi_2)| \langle \xi \rangle \langle \xi \rangle^{-1} \, d\xi_1 \right)^2$$

$$\overset{\text{C.S.}}{\leq} \frac{1}{2\pi} \left(\int_{\mathbb{R}_{\xi_1}} |\hat{u}(\xi_1, \xi_2)|^2 \langle \xi \rangle^2 \, d\xi_1 \right) \left(\int_{\mathbb{R}_{\xi_1}} \langle \xi \rangle^{-2} \, d\xi_1 \right)$$

$$\int_{\mathbb{R}_\xi} \frac{1}{1+\xi_2^2+\xi_1^2} \, d\xi_1 = \left. \frac{\tan^{-1}\left(\frac{\xi}{\sqrt{1+\xi_2^2}} \right)}{\sqrt{1+\xi_2^2}} \right|_{-\infty}^{\infty}$$

$$= \frac{\pi}{\sqrt{1+\xi_2^2}}$$

$$\int_{\mathbb{R}_{\xi_2}} \sqrt{1+\xi_2^2} |\hat{f}(\xi_2)|^2 \, d\xi_2 \leq \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}_{\xi_2}} \int_{\mathbb{R}_{\xi_1}} |\hat{u}(\xi_1, \xi_2)|^2 \langle \xi \rangle^2 \, d\xi_1 \, d\xi_2$$

(Recall that:

$$\int_{-\infty}^{\infty} \frac{1}{a+x^2} \, dx = \frac{\tan^{-1}\left(\frac{x}{\sqrt{a}} \right)}{\sqrt{a}}$$

□

Theorem W.3.

$$T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1}) \qquad \text{is onto}$$

Proof. (n=2)
Given $\hat{f}(\xi_2)$, construct u.

$$\hat{u}(\xi_1, \xi_2) = \frac{\sqrt{\frac{\pi}{2}} \hat{f}(\xi_2) \langle \xi_1 \rangle}{\langle \xi \rangle^2}$$

Given $f \in H^{1/2}(\mathbb{R})$, verify that this u is in $H^1(\mathbb{R}^2)$. □

Remark W.4. ***Poisson Integral Formula***

We are considering harmonic functions in the disk:

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } D = \{x \in \mathbb{R}^2 \mid |x| < 1\} \\ u &= g \quad \text{on } \partial D \quad \text{(Dirichlet boundary condition)} \end{aligned}$$

Solution:

$$u = PI * g$$

Corresponding problem:

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } D \\ \frac{\partial u}{\partial n} &= G \quad \text{on } \partial D \quad \text{(Neumann B.C.)} \end{aligned}$$

X 5-20-11: Fourier Series Revisited

Definition X.1.

For $u \in L^1(\mathbb{T})$,

$$\mathcal{F}(u)(k) = (2\pi)^{-n} \int_{\mathbb{T}^n} u(x) e^{-ik\cdot x}\, dx$$
$$\left[\mathcal{F}^*(\hat{u})\right](x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ik\cdot x}$$

$\mathcal{F} : L^1(\mathbb{T}^n) \to \ell^\infty$

Definition X.2. $\mathfrak{s}$

$$\mathfrak{s} = \mathcal{S}(\mathbb{Z}^n)$$

Rapidly decreasing functions on $\mathbb{Z}^n$, i.e. for every $N \in \mathbb{N}$,

$$\langle k\rangle^n \left|\hat{u}_k\right| \in \ell^\infty$$

$\mathcal{F} : C^\infty(\mathbb{T}^n) \to \mathfrak{s}$

Definition X.3. $\mathcal{D}$, $\mathcal{D}'$

$$\mathcal{D}(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$$
$$\mathcal{D}'(\mathbb{T}^n) = [C^\infty(\mathbb{T}^n)]'$$
$$\mathfrak{s}' = [\mathfrak{s}]'$$

Remark X.4.

$\mathcal{F} : L^2(\mathbb{T}^n) \to \ell^2$
$\mathcal{F}^* : \ell^2 \to L^2(\mathbb{T}^n)$

We define the inner products as

$$(u, v)_{L^2(\mathbb{T}^n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} u(x)\overline{v(x)}\, dx$$
$$(\hat{u}, \hat{v}) = \sum_{k\in\mathbb{Z}^n} \hat{u}_k \overline{\hat{v}_k} \frac{1}{(2\pi)^n} \|u\|_{L^2(\mathbb{T}^n)} = \|\hat{u}\|_{\ell^2}$$

Remark X.5. ***Extension to $\mathcal{D}'(\mathbb{T}^n)$***

$$\begin{aligned}\mathcal{F} :&\mathcal{D}'(\mathbb{T}^n) \to \mathfrak{s}' \\ \mathcal{F}^* :&\mathfrak{s}' \to \mathcal{D}'(\mathbb{T}^n)\end{aligned}$$

Definition X.6. ***Sobolev Spaces on $\mathbb{T}^n$***

$$H^s(\mathbb{T}^n) = \left\{u \in \mathcal{D}'(\mathbb{T}^n) \;\middle|\; \langle k\rangle^s \hat{u} \in \ell^2\right\}, \qquad s \in \mathbb{R}$$

Definition X.7. Λ^s

$$\Lambda^s u = \mathcal{F}^*\left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^s \hat{u}_k e^{ikx}\right)$$

(Where $\langle k\rangle = \sqrt{1+|k|^2}$.)

$$H^s(\mathbb{T}^n) = \Lambda^{-s}L^2(\mathbb{T}^n)$$

This is an isomorphism.

Example X.8.

$$\begin{aligned}\Lambda^2 &= (1-\Delta) \\ \Lambda^{-2} &= (1-\Delta)^{-1} \\ \Lambda^0 &= \mathrm{Id} \\ \Lambda^1 &= \sqrt{1-\Delta}\end{aligned}$$

This is like exponentiating a matrix in linear algebra: $e^{\mathbf{A}}$.

Definition X.9. $H^s(\mathbb{T}^n)$ ***Inner Product***

$$(u,v)_{H^s(\mathbb{T}^n)} = (\Lambda_s u, \Lambda_s v)_{L^2(\mathbb{T}^n)} \qquad s \in \mathbb{R}$$

Remark X.10. ***Poisson Integral Formula***

$$\text{PI}\,(f)(r,\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta}, \qquad r<1$$

Let

$$u(r,\theta) = \text{PI}\,(f)(r,\theta)$$

For example,

$$\begin{aligned} D = \{|x|<1\}, &\qquad \partial D = S^1 = \mathbb{T}^1 \\ -\Delta u = 0 \text{ in } D & \\ u = f \text{ on } \partial D = \mathbb{T}^1 & \end{aligned}$$

Recall from week 2:

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(\phi)}{r^2 - 2r\cos(\theta-\phi)+1}\,d\phi \qquad r<1$$

Given $f \in H^s(\mathbb{T}^1)$, how smooth is u in D?

Remark X.11. ***Recall from Weeks 1 & 2***

$$f \in C(\partial D) \xrightarrow{\text{DCT}} u \in C(\overline{D}) \cap C^\infty(\tilde{D}) \qquad \forall\ \tilde{D} \subset\subset D$$

Remark X.12. ***Δ in 2-D***

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

Then our problem becomes

$$\begin{aligned} -\Delta u &= F \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned}$$

Significance: we are ignoring the cross derivatives, $\frac{\partial^2}{\partial x_1 \partial x_2}$.

$$\begin{aligned} -\Delta u &= F \text{ in } \mathbb{R}^2 \\ u(x) &= \underbrace{\frac{1}{\sqrt{2\pi}}}_{?} \int_{\mathbb{R}^2} \log|x-y| F(y)\, dy \\ u = G * F, &\qquad G = \frac{1}{2\pi} \log|x| \end{aligned}$$

Remark X.13. ***Basic Laplacian Info***

$$\begin{aligned} -\Delta &= \text{div } D \\ L &= \text{div } [A(x)D] \end{aligned}$$

Theorem X.14.

$$\text{PI } : H^{k-1/2}(\mathbb{T}^1) \to H^k(D) \quad \text{continuously}$$

In particular,

$$\|u\|_{H^k(D)} \leq C \|f\|_{H^{k-1/2}(\mathbb{T}^1)}, \qquad k = 0, 1, 2, \ldots$$

Proof. Our clutch formula is

$$u(r, \theta) = \sum_{k \in \mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta}, \qquad r < 1$$

Case 1: $k = 0$
Given $f \in H^{-1/2}(\mathbb{T}^1)$. This means that

$$\sum \langle k \rangle^{-1} |\hat{f}_k|^2 < \infty.$$

Compute $L^2(D)$ norm of $u(r,\theta)$.

$$\|u\|_{L^2(D)}^2 = \int_0^{2\pi}\int_0^1 \left|\sum \hat{f}_k r^{|k|} e^{ik\theta}\right|^2 \underbrace{r\,dr\,d\theta}_{\substack{\text{2-D Lebesgue}\\ \text{measure}}}$$

$$\overset{\text{MCT}}{\leq} 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \int_0^1 r^{2|k|+1}\,dr$$

$$\leq \pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \frac{1}{1+|k|}$$

$$\leq \pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \langle k\rangle^{-1} \qquad \left(\frac{1}{\sqrt{1+|k|^2}} \geq \frac{1}{1+|k|}\right)$$

□

Y 5-23-11

Theorem Y.1. ***Poisson Integral Formula***

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta},\ r<1 \tag{Y.1}$$

$$-\Delta u = 0 \text{ in } D$$
$$u = f \text{ on } \partial D$$

Theorem Y.2.

$$\|u\|_{H^k(D)} \le C\|f\|_{H^{k-1/2}(\partial D)},\ k = 0,1,2,\dots$$

Remark Y.3. ***(Last Time)***

$$\|u\|_{L^2(D)} \le C\|f\|_{H^{-1/2}(\mathbb{T}^1)}\ \forall\ f \in H^{-1/2}(\mathbb{T}^1),\ k=0$$

Today we look at $k > 0$.

Remark Y.4. $k = 1$ ***Case***

Goal: Show

$$\|u\|_{H^1(D)} \le C\|f\|_{H^{1/2}(\mathbb{T}^1)},\ u \in L^2$$

Prove that:

$$\frac{\partial u}{\partial \theta} = u_\theta \in L^2 \quad \text{and} \quad \frac{\partial u}{\partial r} = u_r \in L^2$$

Taking ∂_θ of (Y.1) gives us that

$$u_\theta = \sum_{k\in\mathbb{Z}} \hat{f}_k ikr^{|k|}e^{ik\theta} \tag{Y.2}$$

What's the relationship between $f \in H^{1/2}(\mathbb{T}^2),\ \partial_\theta f \in H^{-1/2}(\mathbb{T}^1)$?

$$\partial_\theta : H^s \to H^{s-1} \text{ continuously (by definition)}$$
$$\|f_\theta\|_{H^{-1/2}(\mathbb{T}^1)} \le C\|f\|_{H^{1/2}(\mathbb{T}^1)}$$

This implies that

$$u_\theta(r,\theta) = \sum_{k\in\mathbb{Z}} (\hat{f}_\theta)_k |r|^k e^{ik\theta}$$
$$v(r,\theta) = \sum_{k\in\mathbb{Z}} \hat{g}_k r^{|k|} e^{ik\theta}$$

From $k=0$:

$$\|u_\theta\|_{L^2(D)} \le c\|f_\theta\|_{H^{-1/2}(\mathbb{T}^1)} \le c\|f\|_{H^{1/2}(\mathbb{T}^1)}$$

We want to know

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_2=0} \overset{?}{=} \frac{\partial f}{\partial x_2}(x_1,0)$$

Two ways to proceed:

1. Keep estimating $\partial_\theta^2, \partial_\theta^3, \ldots$

$$-u_{rr} - \frac{1}{r}u_r = \frac{1}{r^2}u_{\theta\theta}$$
$$-r\partial_r(ru_r) = u_{\theta\theta}$$
$$r^2 u_{rr} + ru_r \in L^2$$

2. $\|ru_r\|_{L^2(D)} = \|u_\theta\|_{L^2(D)}$

$$u_r(r,\theta) = \sum \hat{f}_k |k| r^{|k|-1} e^{ik\theta}$$
$$ru_r(r,\theta) = \sum \hat{f}_k |k| r^{|k|} e^{ik\theta}$$

This has the same L^2 inner product as (Y.2). Thus,

$$\|ru_r\|_{L^2(D)} \le c\|f\|_{H^{1/2}(D)}$$
$$\|u_r\|_{L^2(D)} \overset{?}{\le} c\|f\|_{H^{1/2}(D)}$$

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int \frac{f(\phi)}{r^2 - 2r\cos(\theta-\phi)+1}\,d\phi$$

We can differentiate this as my times as we like in the region $r<\frac{1}{2}$. Thus, $u \in C^\infty\left(B\left(0,\frac{1}{2}\right)\right)$.
Suppose we wanted to solve this problem instead:

$$\begin{aligned} -\Delta w &= h \quad \text{in } D \\ w &= 0 \quad \text{on } \partial D = \mathbb{T}^1 \end{aligned} \quad \overset{f\in H^{1/2}(\mathbb{T}^1)}{\Longleftrightarrow} \quad \begin{aligned} -\Delta u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial D = \mathbb{T}^1 \end{aligned}$$

$$w = u - f \text{ on } \partial D = \mathbb{T}^1$$
$$w = u - \tilde{f} \text{ on } D$$

From the trace theorem we know that $T : H^1(D) \to H^{1/2}(D)$ is a continuous surjection. For every $f \in H^{1/2}(\partial D)$ there exists $\tilde{f} \in H^1(D)$ such that $\|\tilde{f}\|_{H^1(D)} \leq C\|f\|_{H^{1/2}(\partial D)}$.

$$
\begin{aligned}
&f \in H^{1/2}(\partial D) \\
&\tilde{f} \in H^1(D) \\
&u = w + \tilde{f}
\end{aligned}
$$

Then

$$
\begin{aligned}
-\Delta w = \Delta \tilde{f} = h \text{ in } D \\
w = 0 \text{ on } \partial D
\end{aligned}
$$

Let $v \in C_0^\infty(D)$.

$$
\begin{aligned}
0 &= -\int_D (\Delta w + \Delta \tilde{f}) v \, dx \\
&= \int_D Dw \cdot Dv \, dx + \int_D D\tilde{f} \cdot Dv \, dx \\
&= \int_D Dw \cdot Dv \, dx \\
&= -\int_D D\tilde{f} \cdot Dv \, dx \;\forall\; v \in H_0^1(D) \qquad\qquad \overline{C_0^\infty(D)}^{H^1} = H_0^1(D) \\
&= (w, v)_{H^1(D)} = -\int_D D\tilde{f} \cdot Dv \, dx \qquad \text{(Y.3)}
\end{aligned}
$$

Why is it true that $\|Dw\|_{L^2(D)}$ is an $H^1(D)$ equivalent norm for every $w \in H_0^1(D)$? Answer: the Poincare Inequality.

$$
\|w\|_{L^2(D)} \leq C\|Dw\|_{L^2(D)}
$$

From (Y.3), the Riesz Representation Theorem gives us that there exists a unique $w \in H_0^1(D)$.

$$
\begin{aligned}
-\Delta w = h \in H^{-1}(\Omega) \text{ in } \Omega \subset \mathbb{R}^n \text{ open, smooth, bounded} \\
w = g \in H^{1/2}(\partial\Omega) \text{ on } \partial\Omega
\end{aligned}
$$

Better yet, have $h \in C^\infty(\Omega)$ and $g \in C^\infty(\partial\Omega)$.

Z 5-24-11 (Section)

Example Z.1.

Ω open, $\partial\Omega$ is C^1

$$\Delta u = 0$$
$$\frac{\partial u}{\partial n} = g$$

If u_1, u_2 are solutions to the above, then

$$u_1 = u_2 + c$$

Set

$$u = u_1 - u_2$$

Then

$$\Delta u = 0$$
$$\frac{\partial u}{\partial n} = 0$$

$$\int_\Omega u\Delta v + \langle Du, Dv\rangle \, dV = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, dS \qquad (u = v,\ \Delta u = 0)$$
$$\int_\Omega |Du|^2 \, dV = \int_{\partial\Omega} u\frac{\partial u}{\partial n}\, dS$$
$$= 0$$

Thus, $Du = 0$. If Ω is connected, then $u = c$ constant.
Note:

$$\frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = g - g = 0$$

Example Z.2.

$$\Delta u = 0 \text{ in } \Omega = B(0,1)$$
$$\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega = \mathbb{S}^1$$

Example Z.3.

$$\Delta u = 0$$
$$u = f = \sum_k \hat{f}_k e^{ik\theta}$$

Then the solution looks like

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} f_k(r) e^{ik\theta}, \quad |r| < 1$$

Use polar coordinates for Δ, solve the ODE for f_k (using the sum):

$$f_k = r^{|k|}\hat{f}_k$$

Example Z.4.

$$\Delta u = 0 \text{ in } \Omega$$
$$\frac{\partial u}{\partial r} = g \text{ on } \Omega$$

then

$$\sum_{k\in\mathbb{Z}} \hat{f}_k |k| e^{ik\theta} = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ik\theta}$$

We have

$$\hat{g}_k = \hat{f}_k |k|, \quad f \in H^s(\mathbb{S}^1)$$

Define

$$Nf = \sum_{k\in\mathbb{Z}} |k| \hat{f}_k e^{ik\theta}$$

Questions:

1. Is N linear?
2. What is the image of the map?
3. Is the map bounded?
4. More...

$$N : H^s(\mathbb{S}^1) \to H_0^{s-1}(\mathbb{S}^1), \qquad H_0^s(\mathbb{S}^1) = \left\{ g \;\middle|\; \int_{\mathbb{S}^1} g = 0 \right\} \subset H^s(\mathbb{S}^1)$$

This is a closed space because if $g_n \to g$ in $H^s(\mathbb{S}^1)$, $\int_{\mathbb{S}^1} g_n = 0$, then $\int_{\mathbb{S}^1} g = 0$ by DCT (since $g \in L^2(\mathbb{S}^1) \subset L^1(\mathbb{S}^1)$). Also, because

$$|\int_{\mathbb{S}} g| \leq c\|g\|_{L^2} \leq C\|g\|_{H^{s-1}(\mathbb{S}^1)}$$

Also because N is a linear surjective mape:

$$\underbrace{\sum_{k\neq 0} \frac{\hat{g}_k}{|k|} e^{ik\theta}}_{\in H^s(\mathbb{S}^1)} \to \sum \hat{g}_k e^{ik\theta}$$

Is N bounded?

$$\begin{aligned} \|Nf\|^2_{H^{s-1}(\mathbb{S}^1)} &= \sum_k |k|^2 |\hat{f}(k)|^2 (1+|k|^2)^{s-1} \\ &\leq \sum_k (1+|k|^2)^2 |\hat{(}k)|^2 \\ &\leq \|f\|^2_{H^s(\mathbb{S}^1)} \end{aligned}$$
$$\ker N = \{c \;\big|\; c \in \mathbb{C}\} \cong \mathbb{C}$$

N is surjective with coker $N = \{0\} = H^{s-1}(\mathbb{S}^1)/\text{Im } N$. Therefore, N is a *Fredholm operator*, ind $N = 1 - 0 = 1$. Why do we need Fredholm operators? They have a pseudo-inverse:

$$T : x \to y, \quad y \in \text{Im } T$$
$$Tx = y \quad \Rightarrow \quad x = \text{“}T^{-1}\text{”} y$$

Example Z.5.

Find the "inverse" of $N : H^s(\mathbb{S}^1) \to H_0^{s-1}(\mathbb{S}^1)$

$$N^{-1}g = \sum_{k\neq 0} \frac{\hat{g}_k}{|k|} e^{ik\theta}$$

$$Nf = g \tag{Z.1}$$

$$f = c + N^{-1}g \quad \text{general solution to (Z.1)} \tag{Z.2}$$

$$
\begin{aligned}
\hat{g}_k &= \frac{1}{2\pi} \int_{\mathbb{S}^1} g(t) e^{-ikt}\, dt \\
N^{-1}g &= \sum_{k\neq 0} \frac{1}{2\pi} \int_{\mathbb{S}^1} g(t) \frac{3^{-ik(t-\theta)}}{|k|}\, dt \\
&= \int_{\mathbb{S}^1} g(t) \left[\frac{1}{2\pi} \sum_{k\neq 0} \frac{e^{-ik(t-\theta)}}{|k|} \right] dt \\
K(t) &\equiv \sum_{k\neq 0} \frac{e^{ikt}}{|k|} \\
&= \frac{1}{2\pi} \int_{\mathbb{S}^1} g(t) K(\theta - t)\, dt
\end{aligned}
$$

Given a function $g \in H^{1/2}(\mathbb{S}^1) \to$ (pick)$f \in H^{3/2}(\mathbb{S}^1)$.
Neumann problem $\Rightarrow$ Dirichlet problem. $\Rightarrow u \in H^{?}(\Omega)$

Example Z.6.

$$\begin{aligned}
u &= \sum r^{|k|}\hat{f}_k e^{ik\theta} \\
u_{\theta\theta} &= -\sum_k r^{|k|}k^2\hat{f}_k e^{ik\theta} \\
\|u_{\theta\theta}\|^2_{L^2(D)} &= \int_0^{2\pi}\int_0^1 |u_{\theta\theta}|^2 r\,dr\,d\theta \\
&= c\sum_k k^4|\hat{f}_k|^2\int_0^1 r^{2|k|+1}\,dr \\
&= c'\sum_k \frac{k^4|\hat{f}_k|^2}{|k|+1} \\
&\le c'\sum_k \frac{|\hat{f}_k|^2\cdot(1+|k|^2)^2}{|k|+1} \\
&\le c'\sum_k |\hat{f}_k|(1+|k|^2)^{3/2}\frac{(1+|k|^2)^{1/2}}{|k|+1}
\end{aligned}$$

$$\begin{aligned}
\|u\|_{H^2(D)} &\le \tilde{c}\|f\|_{H^{3/2}(\mathbb{S}^1)} \\
\|u_{\theta\theta}\|_{L^2(D)} &\approx \\
|ru_r| &= |u_\theta|
\end{aligned}$$

A 5-25-11

Remark A.1.

$$-\Delta u = 0 \text{ in } \Omega \subset D$$
$$\frac{\partial u}{\partial n} = g \text{ on } \partial D$$

where

$$\frac{\partial u}{\partial n} = Du \cdot \mathbf{n}, \quad \mathbf{n} = \text{ outward unit normal.}$$

1. Ω open, bounded, smooth
   $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism.
2. Ω
   $-\Delta : H^1(\Omega) \to L^2(\Omega)$ is an isomorphism? No.
   - $-\Delta : H^1(\Omega) \setminus \mathbb{R} \to L^2(\Omega)$ is an isomorphism.
3. $\Omega = \mathbb{T}^n$
   $-\Delta : H^1(\mathbb{T}^n) \to H^{-1}(\mathbb{T}^n)$ is an isomorphism?

Note:

$$\langle -\Delta u, v \rangle = \int_\Omega Du \cdot Dv \, dx$$

Example A.2.

$$-\Delta u = 0 \text{ in } \Omega$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$
$$Du \cdot \mathbf{n} = 0$$

$u = 1$ is a solution, dim $(N(-\Delta)) = 1$

Example A.3.

$$u : \Omega \to \mathbb{R}^3$$
$$-\Delta u^i = f^i \text{ in } \Omega$$
$$\sum_{j=1}^{3} \frac{\partial u^i}{\partial x_j} n_j = g^i \text{ on } \partial\Omega$$

What is the null space of this operator?

Remark A.4.

$$L^2(\Omega) = \mathrm{N}(L) \oplus_{L^2} \mathrm{R}(L)$$

(Compactness allows us to not require the closure of R.) What we are trying to do is get rid of the null (N) part and restrict entirely to the R part so that we can invert things.

Whenever you remove the null space, $\mathrm{N}(-\Delta)$, you recover the Poincare inequality:

$$\|u\|_{L^2(\Omega)} \leq C\|Du\|_{L^2(\Omega)}$$

Remark A.5.

$$-\Delta u = 0 \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

We can always solve this problem. And this problem:

$$-\Delta u = h \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Example A.6.

$$
\begin{aligned}
-\Delta u &= 0 \text{ in } \Omega \subset D \\
\frac{\partial u}{\partial n} &= g \text{ on } \partial D
\end{aligned}
$$

When can we solve this problem?

Example A.7.

$$
\begin{aligned}
-\Delta u &= -\text{div } Du \text{ in open set } \Omega \\
\frac{\partial u}{\partial n} &= Du \cdot \mathbf{n} \text{ on } \partial\Omega
\end{aligned}
$$

Recall that

$$
\begin{aligned}
\int_\Omega \text{div } Q\, dx &= \int_{\partial\Omega} Q \cdot \mathbf{n}\, dS \\
\int_\Omega -\Delta u\, dx &= \int_\Omega -\text{div } Du\, dx \\
&= -\int_{\partial\Omega} Du \cdot \mathbf{n}\, dS \\
&= -\int_{\partial\Omega} \frac{\partial u}{\partial n}\, dS
\end{aligned}
$$

Example A.8.

$$-\Delta u = F \text{ in } \Omega = D$$
$$\frac{\partial u}{\partial n} = g \text{ on } \partial D$$

We require that

$$\int_\Omega F(x)\,dx + \int_{\partial\Omega} g(x)\,dS = 0$$

$$-\Delta u = -\text{div } Du \text{ in open set } \Omega$$
$$-\Delta u = F \text{ in } \Omega$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

Solvability condition:

$$\int_\Omega F(x)\cdot \mathbf{1}\,dx = 0$$

In words, we need a function that has 0 average.

Remark A.9.

$$\int_\Omega \text{div } Q\,dx = \int_{\partial\Omega} Q\cdot \mathbf{n}\,dS, \quad \mathbf{n} = \text{ outward normal}$$
$$\int_\Omega \text{curl} Q\,dx = \int_{\partial\Omega} Q\cdot T_\alpha\,dS, \quad T_\alpha = \text{tangent vectors}, \alpha = 1,\dots,n-1$$

Remark A.10.

Laplace operators and the like always have finite-dimensional null spaces.

Remark A.11.

$$-\Delta u = f$$
$$u = 0$$

This operator is an isomorphism. We showed this by studying this problem:

$$\int_\Omega Du \cdot Dv \, dx = \int_\Omega f \cdot v \, dx \ \forall \ v \in H_0^1(\Omega), \ f \in L^2(\Omega), \ u \in H_0^1(\Omega)$$

The reason we can take the Laplacian of an H^1 function is the following theorem:

Theorem A.12.

For $u \in H^2(\Omega)$,

$$-\Delta u = f \text{ a.e. in } \Omega$$
$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$
$$\|u\|_{H^s(\Omega)} \leq C\|f\|_{H^{s-2}(\Omega)}, \quad s \geq 0, \text{ real}$$

B 5-27-11

Problem B.1. ***Homework Problem 1 (6.1)***

$$-\Delta u_f = 0 \text{ in } D$$
$$u_f = f \text{ on } \partial D$$

1. $f \xrightarrow{N} g$

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta}, \quad r < 1$$

2. Compute $\frac{\partial u}{\partial r}(r,\theta)$ in D
3. Take the limit as $r \nearrow 1$, compute the trace of $\frac{\partial u}{\partial r}(1,\theta) = g(\theta)$. (This is not a pointwise limit.)

$$-\Delta u = 0 \text{ in } D$$
$$\frac{\partial u}{\partial r} = g \text{ on } \partial D$$

Dirichlet-to-Neumann:

$$g = Nf \quad \Rightarrow \quad \text{``}g = \left|\frac{\partial}{\partial \theta}\right| f\text{''}$$
$$\hat{g}_k = |k|\hat{f}_k$$

We are given $f \in H^{3/2}(\mathbb{S}^1)$. According to $N = \left|\frac{\partial}{\partial\theta}\right|$, we should require that $g \in H^{1/2}(\mathbb{S})$. We have proven that

$$\|u\|_{H^2(D)} \le C\|f\|_{H^{3/2}(\partial D)} \quad \Rightarrow \quad \frac{\partial u}{\partial r} \in H^1(D)$$

Fixing r close to 1, we can think of

$$\frac{\partial u}{\partial r}(\underbrace{r}_{\text{parameter}}, \theta) \Rightarrow \text{function on } (0, 2\pi)$$

Both

$$\hat{f}_k r^{|k|} e^{ik\theta}, \qquad \hat{f}_k |k| r^{|k|-1} e^{ik\theta}$$

are absolutely summable, since $|k| \le \langle k\rangle^{3/2}\langle k\rangle^{-1/2}$.

$$\sum_{k\in\mathbb{Z}} \hat{f}_k |k| r^{|k|-1} e^{ik\theta} = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \hat{f}_k |k| r^{|k|-1} e^{ik\theta}$$

We bring the derivative through the sum, and the goal is to get uniform bounds on $H^{1/2}(0,2\pi)$. We pass the limit as $r \nearrow 1$ weakly and argue 1) that we can obtain a limit and 2) that this limit is the g that we started with.

$$\left\langle \underbrace{\frac{\partial u}{\partial r}(\theta)}_{\in H^{1/2}}, \underbrace{\phi}_{\in H^{-1/2}} \right\rangle \to \langle G, \phi\rangle$$

B.1 Compensated Compactness

Example B.2.

Suppose we have sequence $(u_j), (v_j)$ that are uniformly bounded in $L^2(\Omega)$.
Question: $u_j \cdot v_j \to ?$

$$\begin{aligned} u_{j_k} &\rightharpoonup u \text{ in } L^2(\Omega) \\ v_{j_k} &\rightharpoonup v \text{ in } L^2(\Omega) \\ u_{j_k} \cdot v_{j_k} &\rightharpoonup u \cdot v \text{ in any topology? No.} \end{aligned}$$

Example B.3.

$$\begin{aligned} u_t + D(u^2) &= f \\ u_t^i + \frac{\partial}{\partial x_j}(u^i u^j) &= f \end{aligned}$$

Smooth out and make nice, e.g. by convolution:

$$\partial_t u_\epsilon + D(u_\epsilon u_\epsilon) = f_\epsilon$$

Now we want to pass the limit as $\epsilon \to 0$. We have that

$$\|u_\epsilon\|_{L^2} \leq M$$

However, we can't pass the weak limit because it doesn't like nonlinearities.

Lemma B.4. ***Div-Curl Lemma***

Suppose $u_j \rightharpoonup u$ in L^2 and $v_j \rightharpoonup v$ in L^2. Suppose curl u_j, div v_j are weakly compact in H^{-1}. Then

$$u_j \cdot v_j \rightharpoonup u \cdot v \text{ in } \mathcal{D}'(\Omega)$$

We are compensating for a lack of compactness by introducing a new structure.
Curl is a measure of rotation
Div is a measure of stretching

Remark B.5. ***Identities from Vector Calculus***

$$\begin{aligned} \text{curl } D\phi &= 0 \quad \phi \text{ scalar} \\ \text{div curl } w &= 0 \quad w \text{ vector} \end{aligned}$$

Remark B.6.

For all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega u_j \cdot v_j \phi \, dx \to \int_\Omega u \cdot v\phi \, dx$$

We have

$$\|v_j\|_{L^2(\Omega)} \leq M \quad \text{uniformly in } j$$

$$\begin{aligned} -\Delta w_j &= v_j \text{ in } \Omega \\ w_j &= 0 \text{ on } \partial\Omega \end{aligned}$$

v_j is bounded in L^2, so

$$\|w_j\|_{H^2(\Omega)} \leq C\|v_j\|_{L^2(\Omega)} \leq CM$$

So $w_{j'} \rightharpoonup w$ in $H^2(\Omega)$. Rellich's theorem tells us that $w_{j'} \to w$ in $H^1(\Omega)$.

$$-\Delta w = \text{curl curl } w - D \text{ div } w$$

$$\begin{aligned} \int_\Omega u_j \cdot v_j \phi \, dx &= \int_\Omega u_j \cdot (-\Delta w_j)\phi \, dx \\ &= \int_\Omega u_j \cdot \text{ curl curl } w_j \phi \, dx - \int_\Omega u_j \cdot D \text{ div } w_j \phi \, dx \\ &= \int_\Omega \underbrace{u_j \cdot \text{ curl}}_{\text{curl } u_j \cdot} \phi \text{ curl } w_j - u_j \cdot D\phi \times \text{ curl } w_j \, dx + \int_\Omega \text{ div } u_j \text{ div } w_j \phi \, dx + \int_\Omega u_j \cdot \text{ div } w_j D\phi \, dx \\ &\to \int_\Omega u \cdot v\phi \, dx \end{aligned}$$

C 5-31-11 (Section)

Remark C.1.

There is an error in the practice problem

$$K(x) = |x|^{1/2}$$
$$u = k * f \quad \Rightarrow \quad u \in W^{1,p}$$

because

$$f = \mathbf{1}_{(a,b)}, \quad -\infty < a < b < \infty$$

If $x > b$ then

$$\begin{aligned} u(x) &= \int_a^b \sqrt{x-y}\,dy = -\frac{2}{3}(x-y)^{3/2}\Big|_a^b \\ &= -\frac{2}{3}(x-b)^{3/2} + \frac{2}{3}(x-a)^{3/2} \end{aligned}$$

and this is not bounded. So if we are working with $W^{1,p}(\mathbb{R})$ then it is not correct, but if we have $W^{1,p}(\Omega)$ with Ω compact then it might make sense. Or if we have $W^{1,p}_{\text{loc}}(\mathbb{R})$. Or replace $|x|^{1/2}$ with $|x|^{-1/2}$.

Problem C.2.

$$u_j \rightharpoonup u \text{ in } W_0^{1,1}(0,1)$$
$$u_j \to u \text{ a.e. TRUE}$$

We have $u \in W_0^{1,1}(0,1)$, $u' \in L^1(0,1)$.

$$\begin{aligned} u_j(x) &= \int_0^x u_j'(t)\,dt \\ &= \int_0^\infty u_j'(t)\mathbf{1}_{[0,x)}(t)\,dt \\ &= \int_0^1 u'(t)\mathbf{1}_{[0,x)}(t)\,dt \end{aligned}$$

Problem C.3.

$$\|\eta_\epsilon * (fg') - f\eta_\epsilon * g'\|_{L^2} \leq C\|f\|_{C_b^1(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}$$

$f \in C_b^1(\mathbb{R}),\ \|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty.$
Hint:

$$\begin{aligned}
\eta_\epsilon * (fg)' &= \eta_\epsilon * (f'g) + \eta_\epsilon * (fg') \\
(\eta_\epsilon * h')(x) &= \int_{\mathbb{R}} \eta_\epsilon(x-y)h'(y)\,dy \\
&= \int_{\mathbb{R}} \frac{\partial}{\partial y}\eta_\epsilon(x-y)h(y)\,dy \qquad\qquad \frac{\partial}{\partial y}\eta_\epsilon(x-y) = -\frac{\partial}{\partial x}\eta_\epsilon(x-y) \\
&= \frac{\partial}{\partial x}\eta_\epsilon * h \\
\eta_\epsilon * g'(x) &= \int \eta_\epsilon(y)g'(x-y)\,dy \\
&= \frac{\partial}{\partial x}\int \eta_\epsilon(y)g(x-y)\,dy \\
&= \frac{\partial}{\partial x}\int \eta_\epsilon(x-y)g(y)\,dy \\
&= (\eta_\epsilon' * g)(x) \\
\left|\eta_\epsilon * (f'g)(x)\right| &= \left|\int \eta_\epsilon(x-y)f'(y)g(y)\,dy\right| \\
&\leq \|f'\|_\infty \left|\int \eta_\epsilon(x-y)g(y)\,dy\right| \\
&\leq \|f'\|_\infty C\|g\|_{L^2(\mathbb{R})} \\
&\leq \|f'\|_\infty \sqrt{\int \eta_\epsilon^2(x-y)\,dy}\|g\|_{L^2} \\
\int |\eta_\epsilon * (f'g)(x)|\,dx &\leq \|f'\|_\infty
\end{aligned}$$

$$h = fg$$

We can estimate the term η_ϵ' by:

$$\begin{aligned}
\|\eta_\epsilon * g'\|_{L^2} &= \|\eta_\epsilon' * g\|_{L^2} \\
&\leq C\|g\|_{L^2}
\end{aligned}$$

And now a double integral term:

$$\begin{aligned}
\int\int \eta_\epsilon^2(x-y)\,dy\,dx &= \int\int \left[\frac{1}{\epsilon}\eta\left(\frac{x-y}{\epsilon}\right)\right]^2\,dx\,dy \\
&= \int\int |\eta(t_1-t_2)|^2\,dt_1\,dt_2
\end{aligned}$$

where

$$x = \frac{t_1}{\epsilon}, \qquad y = \frac{t_2}{\epsilon}, \qquad dxdy = \frac{dt_1dt_2}{\epsilon^2}$$

Problem C.4.

$$u_j = \eta_{j-1} * u, \qquad u \in H^1(\mathbb{R})$$
$$\|u_j'\|_{L^2(\mathbb{R})} \le M$$

Banach-Alaoglu. We have a sequence $u'_{j_k} \to g$ in $L^2(\mathbb{R})$.

$$\left\langle u'_{j_k}, \varphi \right\rangle = -\left\langle u'_{j_k}, \varphi' \right\rangle$$
$$\langle g, \varphi \rangle = -\left\langle u, \varphi' \right\rangle$$

Then $g = u'$, and $u \in H^1(\mathbb{R})$.

$$\|u'\| = \|g\|_{L^2} \le \liminf_j \|u_j'\| \le M$$

Index