Document: Math 207B (Winter 2012)
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Contents

1 1-9-12

1.1 Vibrating String

An elastic string has only tension forces (tangent to the string), e.g. no resistance to bending (rod).

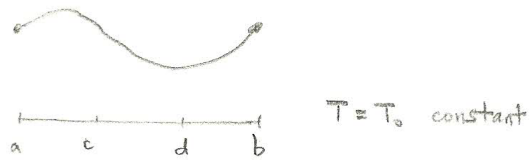
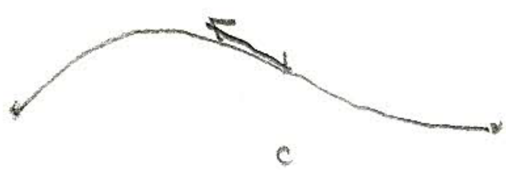


Figure 1: $T = T_0$ (constant)

Straight equilibrium state:

Consider the segment $c \leq x \leq d$. Assume density ρ_0 (mass/unit length).

$x \geq c$ exerts force T on $x \leq c$.



$x \leq c$ exerts force $-T$ on $x \geq c$.

In equilibrium:

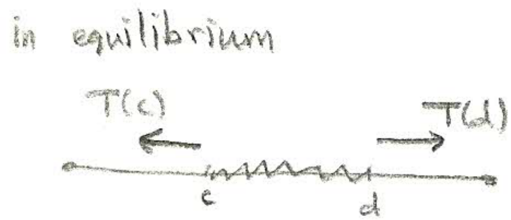
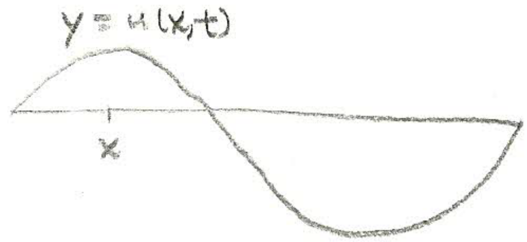
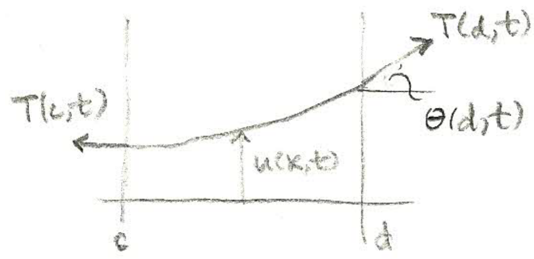


Figure 2: Forces on $c \leq x \leq d$ balance in equilibrium.



Consider small vibrations (transverse). Newton's 2nd Law for section $c \leq x \leq d$:



Vertical direction:

$$\int_c^d \rho_0 u_{tt}\, dx = ma$$

Assume $\rho_0\, ds = \rho_0\, dx$ (mass); assume same because u is small.

$$\int_c^d \rho_0 u_{tt}\, dx = T \sin\theta \big|_{x=c}^{x=d}$$

But $\theta \approx \tan\theta = u_x$ ($\theta \ll 1$), and ignore variations in $T \;\Rightarrow\; T \approx T_0$. Then

$$\int_c^d \rho_0 u_{tt}\, dx = T_0 u_x \big|_{x=c}^{x=d}$$

for any section $c \leq x \leq d$. This is the integral form of <u>conservation of momentum</u> "strong principle" because for any section between c and d:

$$\int_c^d \rho_0 u_{tt}\, dx = \int_c^d T_0 u_{xx}\, dx$$

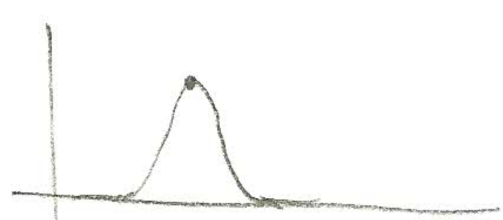
for all c, d (assuming $u(x,t)$ is smooth).

$$\int_c^d (\rho_0 u_{tt} - T_0 u_{xx})\, dx = 0 \qquad \text{(all } a \leq c < d \leq b\text{)}$$

Thus, the integrand is identically zero (assuming the $u_{tt},\ u_{xx}$ are continuous.

$$\rho_0 u_{tt} - T_0 u_{xx} = 0$$

This is *DuBois Reymond's Lemma*.



1-D wave equation:

$$u_{tt} - c_0^2 u_{xx} = 0$$
$$c_0^2 = \frac{T_0}{\rho_0}$$

2-D analog: drum.

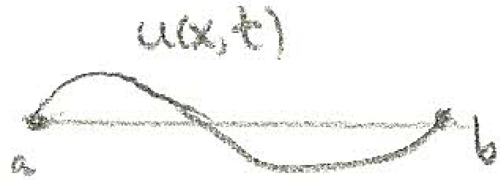
Check dimensions:

$$[c_0^2] = \frac{[T_0]}{[\rho_0]} = \frac{ML/T^2}{M/L} = \frac{L^2}{T^2}$$
$$[c_0] = \frac{L}{T} \qquad \text{(velocity)}$$
$$c_0 = \text{ transverse wave speed}$$

Heavier strings ⇒ waves propogate slower.

Initial conditions: u and u_t
Boundary conditions: one on each end (u or u_x)

1.2 Initial-Boundary Value Problem



$$u_{tt} - c^2 u_{xx} = 0 \qquad \text{PDE}$$
$$u(a,t) = 0,\ u(b,t) = 0 \qquad \text{BC's (Dirichlet)}$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x) \qquad \text{IC's (initial displacement } f\text{, velocity } g\text{)}$$

2 1-11-12

2.1 Vibrating String

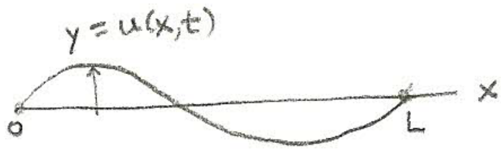


Figure 3: $c_0^2 = \frac{T_0}{\rho_0}$.

$$\begin{aligned} u_{tt} - c_0^2 u_{xx} &= 0 \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

Look for time-periodic, separated solutions of the form

$$u(x,t) = e^{-i\omega t} v(x)$$

where $\omega \in \mathbb{R}$ is the frequency and $v(x)$ is a real-valued function.
$\rightarrow$ separate dependence on time and space.

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$

The real and imaginary parts of a complex solution are themselves solutions (because it is a linear ODE with real coefficients).

Nonlinear equation:
You might try

$$\begin{aligned} u(x,t) &= e^{-i\omega t} v(x) + e^{i\omega t} v(x) \\ \Rightarrow \quad -\omega^2 e^{-i\omega t} v - c_0^2 e^{-i\omega t} v'' &= 0 \end{aligned}$$

$$\begin{aligned} -v'' &= \lambda v, \qquad \lambda = \frac{\omega^2}{c_0^2} \\ v(0) &= 0 \\ v(L) &= 0 \end{aligned}$$

Sturm-Liouville Eigenvalue Problem:
Find eigenvalues λ for which we have nonzero functions $v(k)$.

Claim: We only have nonzero solutions for $\lambda > 0$, say $\lambda = k^2$.

$$-v'' = k^2 v$$

Solution:

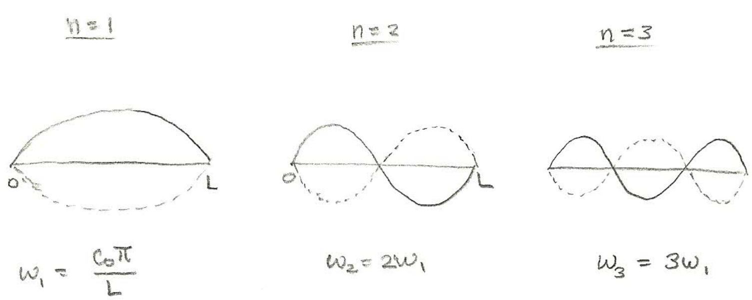
$$v(x) = \cos kx \quad \text{or} \quad v(x) = \sin kx$$

Impose boundary conditions:

$$\begin{aligned}
v(0) &= c_1 = 0 \\
v(L) &= c_2 \sin kL = 0 \quad \Rightarrow \quad kL = n\pi,\ n = 1,2,3,\ldots \ \in \mathbb{N} \\
\lambda &= \lambda_n,\ \lambda_n = \left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{N} \\
v &= v_n,\ v_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\
\omega^2 &= c_0^2 \lambda \\
\omega_n &= \pm c_0 \left(\frac{n\pi}{L}\right)
\end{aligned}$$

The solutions of the wave equation are:

$$\begin{aligned}
u(x,t) &= e^{-i\omega_n t} \sin\left(\frac{n\pi x}{L}\right) \\
&= \begin{cases} \cos(\omega_n t) \sin\left(\frac{n\pi x}{L}\right) \\ \sin(\omega_n t) \sin\left(\frac{n\pi x}{L}\right) \end{cases}
\end{aligned}$$



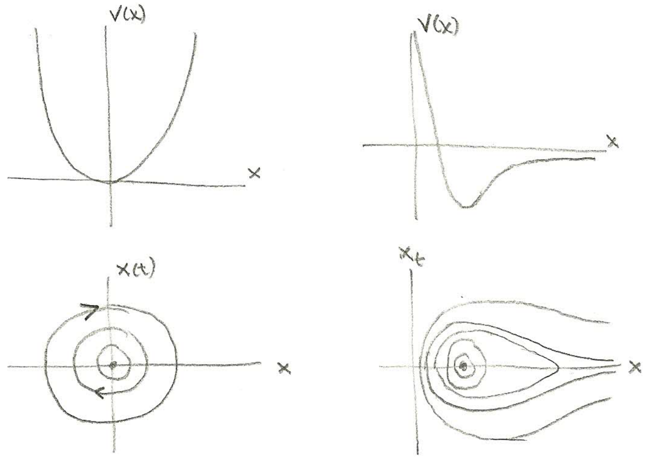
The nth eigenfunction has $n-1$ zeros in $(0, L)$.

2.2 Quantum Mechanics

A single particle of mass m moving in one space dimension with potential $V(x)$.

Classical mechanics: position $x(t)$ satisfies

$$\begin{aligned}
m x_{tt} &= -V'(x) \\
f(x) &= -V'(x)
\end{aligned}$$



In quantum mechanics, we describe the particle by the complex-valued wavefunction $\Psi(x,t)$, where

$$(\text{probability of finding particle } m \text{ in } a \le x \le b) = \int_a^b |\Psi|^2\, dx$$

and Ψ is normalized so that $\int_{-\infty}^{\infty} |\Psi|^2\, dx = 1$. We have the *Schrödinger equation*:

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V(x)\Psi.$$

3 1-13-12

Office Hours: MWF 2:30-3:30

3.1 Schrödinger Equation

Particle of mass m moving in potential $V(x)$.

- Classical equation for position $x(t)$:
$$mx_{tt} = -V'(x)$$
- Quantum description: wavefunction $\Psi(x,t)$ (complex-valued)
$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V(x)\Psi$$
where $\hbar =$ Planck's constant and $h = 2\pi\hbar$.
$$\begin{aligned} [\hbar] &= \text{ Energy} \times \text{Time} \\ &= \text{ Momentum} \times \text{Length} \\ &= \frac{ML^2}{T} \qquad \text{called an action} \\ \hbar &\approx 10^{-34}\text{ J}\cdot\text{s} \end{aligned}$$

Look for separable solutions:
$$\Psi(x,t) = e^{-iEt/\hbar}\phi(x)$$
where E is a real constant and $\phi(x)$ is a real-valued function.
$$\begin{aligned} |\Psi|^2 &= |\phi(x)|^2 \\ &= \text{ stationary probability density} \end{aligned}$$

- E: energy state
- Stationary State: probability density is constant even though Ψ is a function of t

Plug separated Ψ into the Schrödinger equation:
$$\begin{aligned} -\frac{\hbar^2}{2m}\phi'' + V(x)\phi = E\phi \\ -\phi'' + q(x)\phi = \lambda\phi, \qquad q(x) = \frac{2m}{\hbar^2}V(x), \quad \lambda = \frac{2mE}{\hbar^2} \end{aligned}$$

Linear in ϕ, not constant coefficients, second order.
$\Rightarrow$ Cannot analytically solve this! In general, we can't write down explicit solutions.

3.2 Particle in a Box

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

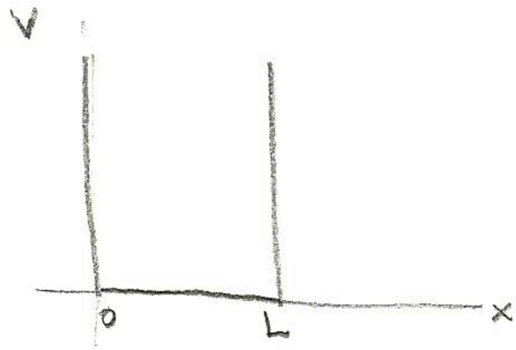


Figure 4: The particle will never be outside this interval.

Classical solution: the particle just bounces back and forth.

$\Psi = 0$ outside the box. Ψ is continuous, so it is 0 at the ends.

$$\begin{cases} -\phi'' = \lambda\phi \\ \phi(0) = \phi(L) = 0 \end{cases} \qquad 0 < x < L$$

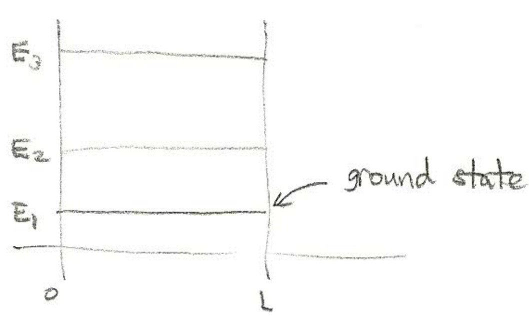
This is the wave equation!

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \dots$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Really should have a constant t_∞ so that $\int |\Psi|^2 \, dx = 1$.

$$E_n = \frac{\hbar^2 \lambda_n}{2m} = \frac{\hbar^2}{2m} \cdot \frac{n^2\pi^2}{L^2}$$
$$E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2, \qquad n = 1, 2, 3, \dots$$

Energy levels of the system.



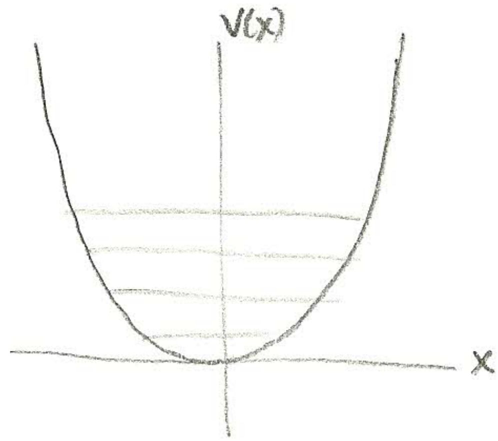
$\Rightarrow$ Energy is discrete, not continuous, with a non-zero ground state energy level!

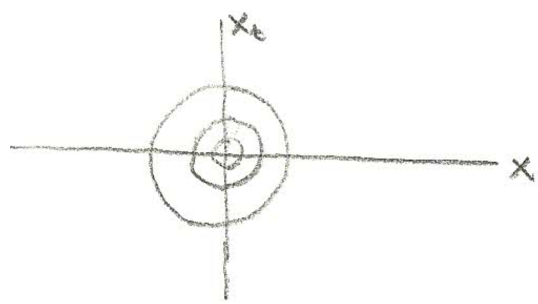
$n = 0$ means $\phi = 0 \Rightarrow$ zero probability of finding the particle.

3.3 Simple Harmonic Oscillator

$$V(x) = \frac{1}{2}kx^2$$
$$mx_{tt} + kx = 0$$
$$x_{tt} + \omega_0^2 x = 0, \qquad \omega_0^2 = \frac{k}{m}$$
$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$\begin{cases} -\phi'' + cx^2\phi = \lambda\phi & c = \frac{mk}{\hbar^2} \\ \phi(x) \to 0 \text{ as } |x| \to \infty & \lambda = \frac{2mE}{\hbar^2} \end{cases}$$





This is an example of a singular Sturm-Liouville problem (on an infinite interval).
⇒ We can solve this exactly.

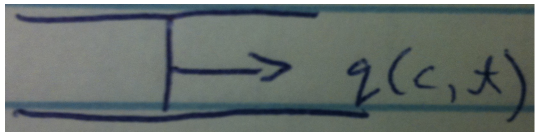
$$\lambda_n = \hbar\omega_0\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots$$
$$\phi_n(x) = H_n(x)e^{-ax^2/2}$$

Equally spaced eigenvalues.

4 1-18-12

4.1 Heat Flow in a Rod

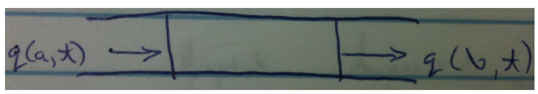
$$\begin{aligned} e(x,t) &= \text{ thermal energy/unit length} \\ q(x,t) &= \text{ heat flux} \\ u(x,t) &= \text{ temperature at point } x \text{ at time } t \end{aligned}$$



$$\begin{aligned} q(c,t) &= \text{ rate at which thermal energy flows from } x < c \text{ to } x > c \\ f(x,t) &= \text{ heat source/unit length} \end{aligned}$$

Conservation of heat energy in section $a < x < b$.

$$\frac{d}{dt}\int_a^b e\,dx = -q(b,t) + q(a,t) + \int_a^b f\,dx$$



This is an integral form of conservation of energy. We want to write this as a PDE.

$$\begin{aligned} \int_a^b e_t\,dx &= -\int_a^b q_x\,dx + \int_a^b f\,dx \\ &= \int_a^b (e_t + q_x - f)\,dx = 0 \qquad \forall\ [a,b] \end{aligned}$$

Provided the integrand is continuous, it follows that

$$e_t + q_x = f \qquad \text{(du Bois-Reymond Lemma)}$$

Conservation (or balance, if $f \neq 0$) of energy (differential form).

Constitutive relations are needed for e, q, f in order to solve. Let $u =$ temperature.

1. $e = cu$, where $c =$ thermal capacity. Let's work with the nonuniform case: $c = c(x)$.
2. $q = -\kappa u_x$ (negative because heat flows from hot to cold), $\kappa =$ thermal conductivity
3. $f = -\gamma u$

From these relations, we get

$$\begin{aligned} cu_t - (\kappa u_x)_x &= -\gamma u \\ cu_t &= (\kappa u_x)_x - \gamma u \end{aligned}$$

A heat or diffusion equation. If c, κ are constant (uniform rod) and $\gamma = 0$, then

$$u_t = \nu u_{xx}$$

where $\nu = \frac{\kappa}{c}$, $[\nu] = \frac{L^2}{T}$. Characteristic length scale: $L \sim \sqrt{\nu T}$.

Since it is first order in time, we need 1 initial condition.

4.2 Boundary Conditions

1. Fixed temperature: $u(0,t) = u(L,t) = 0$ (Dirichlet BCs)

2. Insulated: $q(0) = q(L) = 0 \Rightarrow u_x(0,t) = u_x(L,t) = 0$ (Neumann BCs)

3. Newton's Law of Cooling: $q \propto u$

$$-\kappa u_x = -\alpha u$$
$$u_x = \frac{-\alpha}{\kappa} u$$

   Thus,

$$u_x(0,t) + \alpha u(0,t) = 0$$
$$u_x(L,t) + \beta u(L,t) = 0$$

   (Mixed or Robin BCs)

4. Periodic: $u(0,t) = u(L,t),\ u_x(0,t) = u_x(L,t)$ (not separated like the other 3 BCs)

$$u_t = (\kappa u_x)_x - \gamma u$$

Look for separated solutions:

$$u(x,t) = e^{-\lambda t} v(x)$$
$$-\lambda c v = (\kappa v')' - \gamma v$$
$$-(\kappa v')' + \gamma v = \lambda c v, \qquad 0 < x < L$$

Let's consider the Dirichlet boundary conditions: $v(0) = L(0) = 0$. This is a Sturm-Liouville eigenvalue problem. λ is the rate at which the corresponding eigenfunction decays in time.

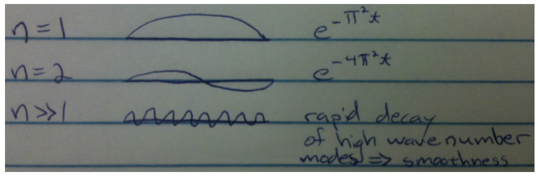
Now take κ, c constant and $\gamma = 0$. After nondimensionalization (rescaling), we can set all the constants to 1.

$$u_t = u_{xx}, \qquad 0 < x < 1$$
$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$$
$$u(x,t) = e^{-\lambda t} v(x)$$
$$\begin{cases} -v'' = \lambda v \qquad 0 < x < 1 \\ v(0) = v(1) = 0 \end{cases}$$
$$v_n(x) = \sin(n\pi x), \qquad \lambda_n = n^2\pi^2, \quad n = 1, 2, 3, \ldots$$

Our separated solutions look like:

$$u(x,t) = e^{-n^2\pi^2 t} \sin(n\pi x)$$

(Note: if we had cosines then we would want to consider $n = 0$.)



General solution of the heat equation:

$$
\begin{aligned}
u_t &= u_{xx}, \qquad 0 < x < 1 \\
u(0,t) &= 0, \qquad u(1,t) = 1 \\
u(x,0) &= f(x) \\
u(x,t) &= \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x) \\
f(x) &= \sum_{n=1}^{\infty} c_n \sin(n\pi x)
\end{aligned}
$$

Where the c_i's are chosen to satisfy this last equation.

5 1-20-12

5.1 Sturm-Liouville Eigenvalue Problems (EVP)

$$
\begin{aligned}
-(pu')' + qu &= \lambda u, \qquad a < x < b \qquad (5.1)\\
\alpha_1 u(a) + \alpha_2 u'(a) &= 0\\
\beta_1 u(b) + \beta_2 u'(b) &= 0
\end{aligned}
$$

Assume p, p', q are continuous functions on $a \le x \le b$. We want to find eigenvalues $\lambda \in \mathbb{R}$ (we will see that λ must be real) such that (5.1) has nonzero solutions u (eigenfunctions). For regular Sturm-Liouville EVP, we get an infinite sequence of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots,\ \lambda_n \to \infty$ as $n \to \infty$, and a complete set of orthogonal eigenfunctions $u_n(x)$.

Claim: we can write every function f as a linear combination of these eigenfunctions,

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x).$$

$$
\begin{aligned}
L &= -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \qquad \text{(Sturm-Liouville operator)}\\
Lu &= -(pu')' + qu
\end{aligned}
$$

Sturm-Liouville Eigenvalue Problem (SL EVP): look for scalars λ such that

$$
\begin{aligned}
Lu &= \lambda u\\
B(u) &= 0 \qquad \text{(BC's)}
\end{aligned}
$$

Definition 5.1. ***Green's Identity***

Let $u, v : [a,b] \to \mathbb{R},\ u, v \in C^2[a,b]$ (twice continuously differentiable on $[a,b]$.

$$
\begin{aligned}
\int_a^b uLv\,dx &= \int_a^b u\left\{-(pv')' + qv\right\} dx\\
&\overset{\text{IBP}}{=} \int_a^b \left\{pu'v' + quv\right\} dx - [puv']\big|_a^b\\
&\overset{\text{IBP}}{=} \int_a^b \underbrace{\left\{-(pu')'v + quv\right\}}_{Lu} v\,dx + [pu'v - puv']\big|_a^b
\end{aligned}
$$

$$\int_a^b [uLv - vLu]\,dx = [p(u'v - uv')]\big|_a^b$$

Let $L^2(a,b) =$ the space of functions $f : [a,b] \to \mathbb{R}$ such that

$$\int_a^b |f|^2\,dx < \infty$$

We define an inner product

$$(f,g) = \int_a^b f(x)g(x)\,dx,$$
$$\|f\| = \left(\int_a^b |f|^2\,dx\right)^{1/2}$$
$$(u, Lv) = (Lu, v) + [p(u'v - uv')]\big|_a^b$$

This last equality tells us that L is *formally self-adjoint*.

Suppose $u(a) = u(b) = 0$ (Dirichlet BC's). Then

$$\begin{aligned}[p(u'v - uv')]\big|_a^b &= [pu'v]\big|_a^b \\ &= p(b)u'(b)v(b) - p(a)u'(a)v(a)\end{aligned}$$

The boundary terms vanish for all such u if and only if $v(a) = v(b) = 0$. In that case, we say that the Dirichlet BC's are self-adjoint. If u, v both satisfy Dirichlet BC's, then

$$(u, Lv) = (Lu, v)\,.$$

Suppose

$$\begin{aligned}Lu &= \lambda u \\ u(a) &= u(b) = 0 \\ Lv &= \mu v \\ v(a) &= v(b) = 0\end{aligned}$$

$\lambda, \mu \in \mathbb{R},\ \lambda \neq \mu$.

$$\begin{aligned}(u, Lv) &= (Lu, v) \\ (u, \mu v) &= (\lambda u, v) \\ \mu\,(u,v) &= \lambda\,(u,v) \\ (u,v) &= 0 \qquad \text{if } \lambda \neq \mu\end{aligned}$$

We say that u and v are *orthogonal*, and we write $u \perp v$. Thus, we have the following theorem:

Theorem 5.2.

Eigenfunctions of a Sturm-Liouville EVP with distinct eigenvalues are orthogonal.

Example 5.3.

$$
\begin{aligned}
L &= -\frac{d^2}{dx^2} \\
-u'' &= \lambda u, \qquad 0 < x < 1 \\
u(0) &= u(1) = 0
\end{aligned}
$$

Solution:

$$
\begin{aligned}
\lambda_n &= n^2\pi^2, \qquad n = 1, 2, \ldots \\
u_n &= \sin(n\pi x)
\end{aligned}
$$

Let's look at inner products of eigenfunctions:

$$
\begin{aligned}
(u_n, u_m) &= \int_0^1 \sin(n\pi x)\sin(m\pi x)\, dx \\
&= \frac{1}{2}\int_0^1 \cos[(n-m)\pi x] - \cos[(n+m)\pi x]\, dx \\
&= 0
\end{aligned}
$$

Thus, the eigenfunctions are orthogonal.

All eigenvalues of the SL EVP are real.
For complex-valued functions, $f, g : [a, b] \to \mathbb{C}$, we define the inner product as

$$
\begin{aligned}
(f, g) &= \int_a^b f(x)\overline{g(x)}\, dx \\
\|f\| &= \left(\int_a^b |f|^2\, dx\right)^{1/2} \\
\|f\|^2 &= (f, f)
\end{aligned}
$$

Thus, if c is a complex constant, then

$$
\begin{aligned}
(cf, g) &= c\,(f, g) \\
(f, cg) &= \overline{c}\,(f, g)
\end{aligned}
$$

For the Sturm-Liouville problem, assume p, q are real-valued.

$$
\begin{aligned}
(u, Lv) &= \int_a^b u\overline{[-p(v')' + q]}\, dx \\
&= \int_a^b u[-(p\overline{v}')' + q\overline{v}]\, dx \\
&= (Lu, v) + [p(u\overline{v}' - u'\overline{v})]\big|_a^b
\end{aligned}
$$

If $u(a) = u(b) = 0$ and $v(a) = v(b) = 0$ (Dirichlet BC's), then

$$
(u, Lv) = (Lu, v)\,.
$$

Suppose $Lu = \lambda u$, where $\lambda \in \mathbb{C}$ and $u \neq 0$.

$$\begin{aligned}
(u, Lu) &= (Lu, u) \\
(u, \lambda u) &= (\lambda u, u) \\
\overline{\lambda}\,(u, u) &= \lambda \underbrace{(u, u)}_{=\|u\|^2 \neq 0} \\
\overline{\lambda} &= \lambda
\end{aligned}$$

Thus, $\lambda \in \mathbb{R}$.

Theorem 5.4.

Every eigenvalue λ of a SL EVP problem is real.

So our 2 main results for the SL EVP problem are:

1. Eigenfunctions are orthogonal.
2. Eigenvalues are real.

6 1-23-12

6.1 Orthogonal Expansions

$L^2(a,b)$ = the space of (Lebesgue integrable) functions $f : (a,b) \to \mathbb{C}$ such that

$$\int_a^b |f|^2 \, dx < \infty.$$

This is a Hilbert space with the inner product

$$(f,g) = \int_a^b f(x)g(x)\, dx.$$

(This is the convention used by Logan. He discusses this is section 4.1.)

$$\begin{aligned} \|f\| &= (f,f)^{1/2} \\ &= \left(\int_a^b |f|^2 \, dx \right)^{1/2} \end{aligned}$$

We say that f, g are *orthogonal* if $(f,g) = 0$. A set of (linearly independent) functions $\{\phi_1, \phi_2, \phi_3, \ldots\}$ is a complete orthogonal set in $L^2(a,b)$ if every function $f \in L^2(a,b)$ can be expanded uniquely as

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$\lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} c_n \phi_n \right\| = 0.$$

Equivalently,

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} c_n \phi(x) \right|^2 dx \to 0 \qquad \text{as } N \to \infty.$$

Note that

$$\begin{aligned} (f, \phi_n) &= \left(\sum_{k=1}^{\infty} c_k \phi_k, \phi_n \right) \\ &= \sum_{k=1}^{\infty} c_k \left(\phi_k, \phi_n \right) \\ &= c_n \|\phi_n\|^2 \\ c_n &= \frac{(f, \phi_n)}{\|\phi_n\|^2} = \frac{\int_a^b f(x)\overline{\phi_n(x)}\, dx}{\int_a^b |\phi_n(x)|^2 \, dx} \end{aligned}$$

For an orthonormal set $\{\phi_1, \phi_2, \ldots\}$,

$$c_n = \int_a^b f(x)\overline{\phi_n(x)}\, dx$$

6.2 2 Inequalities

Theorem 6.1. ***Cauchy-Schwarz Inequality***

$$|(f,g)| \leq \|f\| \cdot \|g\|$$
$$\left|\int_a^b f\overline{g}\,dx\right| \leq \left(\int_a^b |f|^2\,dx\right)^{1/2} \left(\int_a^b |g|^2\,dx\right)^{1/2}$$

Theorem 6.2. ***Parseval's Inequality***

$$\begin{aligned}\|f\|^2 &= (f,f) \\ &= \left(\sum_{n=1}^{\infty} c_n\phi_n, \sum_{k=1}^{\infty} c_k\phi_k\right) \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} c_n\overline{c_k}\,(\phi_n,\phi_k) \\ &= \sum_{n=1}^{\infty} |c_n|^2\|\phi_n\|^2\end{aligned}$$

The L^2 norm often has an interpretation as energy.

6.3 Sturm-Liouville Problems

$$\begin{aligned}Lu &= \lambda u, \qquad a < x < b \\ B(u) &= 0 \qquad \text{(BC's)} \\ Lu &= -(pu')' + qu \\ &= -pu'' - p'u' + qu\end{aligned}$$

where $p(x), q(x)$ are given coefficient functions.

Boundary conditions:
Either

1. Separated BC's: $\alpha_1 u(a) + \alpha_2 u'(a) = 0,\ \beta_1 u(b) + \beta_2 u'(b) = 0$, where α_1, α_2 and β_1, β_2 are not both zero.

2. Periodic BC's: $u(a) = u(b),\ u'(a) = u'(b)$

We say that this is a *regular Sturm-Liouville EVP* if

1. p, p', q are continuous on $[a, b]$

2. $[a, b]$ is a finite interval

3. $p > 0$ for all $x \in [a, b]$

   - If p has a zero in the interval, the system changes from second order to first order $\Rightarrow$ singular behavior.
   - If $p < 0$ for all $x \in [a, b]$ then we can multiply through the equation by -1 and change the sign; the point is it must be nonzero and it can't change sign.

With this L and B, the problem is self-adjoint:

$$\int_a^b (uLv - vLu)\,dx = 0 \qquad \forall\, u, v \in C^2[a, b],\ Bu = 0,\ Bv = 0$$

Theorem 6.3.

The eigenvalues $-\infty < \lambda_1 \le \lambda_2 \le \cdots \le \cdots \le \lambda_n \le \cdots$ of the regular SLP EV Problem are real, and in the case of separated BC's they are distinct (i.e. strict inequality), and $\lambda_n \to \infty$ as $n \to \infty$. Eigenfunctions with different eigenvalues are orthogonal, and the eigenfunctions $\{u_1, u_2, \ldots, u_n, \ldots\}$ are complete in $L^2(a, b)$.

Example 6.4.

$$\begin{aligned} -u'' &= \lambda u, \qquad 0 < x < 1 \\ u(0) &= u(1) = 0 \end{aligned}$$

$$\begin{aligned} \lambda_n &= n^2\pi^2, \qquad n = 1, 2, \ldots \\ u_n(x) &= \sin(n\pi x) \end{aligned}$$

The claim is that we can write an arbitrary function f in terms of these eigenfunctions.

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} c_n \sin(n\pi x) \\ \frac{1}{2} &= \int_0^1 \sin^2(n\pi x)\,dx \\ c_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx \end{aligned}$$

7 1-25-12

7.1 Sturm-Liouville EVP

$$-(pu')' + qu = \lambda u, \qquad a < x < b$$
$$\alpha_1 u(a) + \alpha_2 u'(a) = 0$$
$$\beta_1 u(b) + \beta_2 u'(b) = 0$$

Separated BC's (α_1, α_2 and β_1, β_2 not both zero).

Definition 7.1. ***Regular***

A SL EVP is regular if

1. $[a, b]$ is a finite interval
2. p, p', q are continuous on $[a, b]$
3. $p(x) > 0,\ a \le x \le b$ (including endpoints)

Theorem 7.2.

The eigenvalues of a regular SL-EVP are real and they form an infinite increasing sequence $-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ (with no accumulation points) such that $\lambda_n \to \infty$ as $n \to \infty$. The eigenvalues are simple (one-dimensional eigenspace) and the corresponding (normalized) eigenfunctions $\{u_1(x), u_2(x), \ldots, u_n(x), \ldots\}$ are orthogonal in $L^2(a, b)$ and complete.

Theorem 7.3. ***Oscillation Theorem***

For the regular SL-EVP with separated BC's, then the nth eigenfunction $u_n(x)$ has exactly $n - 1$ zeros in the (open) interval (a, b). Moreover, the zeros of the $(n + 1)$th eigenfunction $u_{n+1}(x)$ lie between the zeros of $u_n(x)$ or the endpoints a, b.

Example 7.4. ***Dirichlet***

$$-u'' = \lambda u, \qquad 0 < x < 1, \quad L = -\frac{d^2}{dx^2}, \quad p = 1,\ q = 0$$
$$u(0) = 0, \quad u(1) = 0$$
$$\lambda_n = n^2\pi^2, \qquad n = 1, 2, 3, \dots$$
$$u_n(x) = \sin(n\pi x)$$
$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} \frac{1}{2} & n = m \\ 0 & n \neq m \end{cases}$$

Fourier sine-series. $f \in L^2(0,1)$,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$
$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

Example 7.5. ***Neumann***

$$-u'' = \lambda u, \qquad 0 < x < 1, \quad L = -\frac{d^2}{dx^2}, \quad p = 1,\ q = 0$$
$$u'(0) = 0, \quad u'(1) = 0$$
$$\lambda_n = n^2\pi^2, \qquad n = 0, 1, 2, \dots$$
$$u_n(x) = \cos(n\pi x)$$
$$\int_0^1 1 \cdot \cos(n\pi x)\,dx = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}$$
$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} \frac{1}{2} & n = m \\ 0 & n \neq m \end{cases}, \qquad n, m \geq 1$$
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$
$$a_0 = \int_0^1 f(x)\,dx$$
$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx, \qquad n \geq 1$$

$u_0(x) = 1,\ u_1(x) = \cos(\pi x)$. u_1 has 1 zero in (a,b), *but* u_1 is actually the second eigenfunction, so the Oscillation Theorem still holds.

Example 7.6. ***Periodic***

$$
\begin{aligned}
-u'' &= \lambda u, \qquad 0 < x < 2\pi, \quad L = -\frac{d^2}{dx^2}, \quad p = 1,\ q = 0 \\
u(0) &= u(2\pi), \quad u'(0) = u'(2\pi) \\
\lambda_n &= n^2, \qquad n \in \mathbb{Z},\ -\infty < n < \infty \\
u_n(x) &= e^{inx}
\end{aligned}
$$

λ_0 is simple: $u_0(x) = 1$.
$\lambda_n = n^2$ has 2 independent eigenfunctions, e^{inx} and e^{-inx}.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

For $f \in L^2(0, 2\pi)$, it has Fourier series

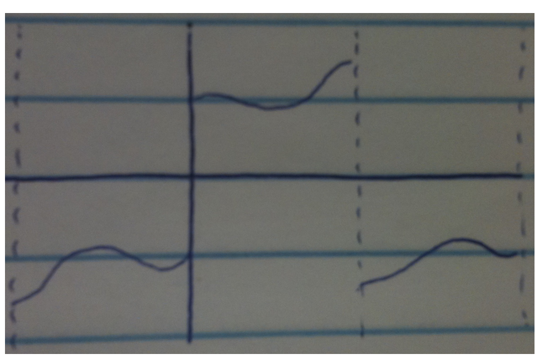
$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
c_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx
\end{aligned}
$$

If f is real-valued, then $c_{-n} = \overline{c_n}$.

7.2 Sine and Cosine Series

Let's take the Fourier sine series:

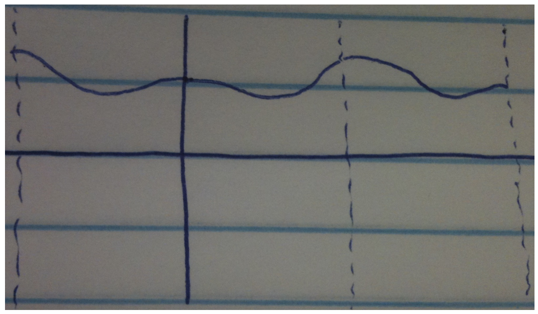
$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$



This is a Fourier series of the odd, 2-periodic extension of f. We get the *Gibbs phenomenon* at the jump discontinuity. The spike doesn't get smaller (in magnitude) as we include more terms in the Fourier series, but it does get narrower, so we still get L^2 convergence.

Now we look at the cosine series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$



The cosine series won't have a jump discontinuity, but it could have a corner. It will typically converge faster than the sine series.

8 1-27-12

8.1 Separation of Variables (Again)

Heat Equation/BVP

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < 1 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

Solutions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

Initial condition at $t = 0$:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \sin(n\pi x) \\ c_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx \end{aligned}$$

Remarks

1. The solution is a smooth function of x for all $t > 0$ (because its Fourier coefficients, $c_n e^{-n^2\pi^2 t}$, decay exponentially fast as $n \to \infty$).

   $$\partial_x^{2k} u(x,t) = (-1)^k \sum_{n=1}^{\infty} (n\pi)^{2k} e^{-n^2\pi^2 t} \sin(n\pi x)$$

   Diffusion immediately damps out the high frequency modes.

2. Irreversible (can't continue backwards in time in general). $\Rightarrow$ This would entail exponentially *growing* Fourier coefficients.

3. As $t \to \infty$, $u(x,t) \to 0$. For large t, $u(x,t) \sim c_1 e^{-\pi^2 t} \sin(\pi x)$ (assuming $c_1 \neq 0$).

We have a "*spectral gap*" here: the first eigenvalue is separated from higher eigenvalues, and thus the higher eigenvalues damp out.

Insulated Rod

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < 1 \\ u_x(0,t) &= u_x(1,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

Solution:

$$\begin{aligned} u(x,t) &= c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \cos(n\pi x) \\ c_0 &= \int_0^1 f(x)\,dx \\ c_n &= 2\int_0^1 f(x)\cos(n\pi x)\,dx \end{aligned}$$

The same comments about smoothing and irreversibility apply here.

As $t \to \infty$, $u(x,t) \to c_0 = \int_0^1 f(x)\,dx$. Thus, thermal energy is conserved.

Conservation of Energy

$$\begin{aligned} u_t &= u_{xx} \\ \int_0^1 u_t\,dx &= \int_0^1 u_{xx}\,dx \\ \frac{d}{dt}\left(\int_0^1 u\,dx\right) &= u_x\big|_0^1 = 0 \\ \int_0^1 u(x,t)\,dx &= \text{ constant} \end{aligned}$$

Schrödinger Equation

$$\begin{aligned} iu_t &= -u_{xx} + q(x)u, \qquad 0 < x < 1 \\ u(0,t) &= 0 = u(1,t) \\ u(x,0) &= f(x) \end{aligned}$$

Solution

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n e^{-i\lambda_n t}\phi_n(x) \\ -\phi_n'' + q(x)\phi_n &= \lambda_n\phi_n, \qquad n = 1,2,\dots \\ \int_0^1 \phi_n^2\,dx &= 1 \qquad \phi_n\text{'s are assumed to be real} \\ c_n &= \int_0^1 f(x)\phi_n(x)\,dx \end{aligned}$$

Remarks

1. There is no decay. In fact,
$$\int_0^1 |u|^2\,dx = \text{ constant}$$
2. Oscillation in time (almost periodic)
3. No smoothing. If you stick in a jump discontinuity you get oscillatory behavior.

8.2 Green's Functions

(Section 4.4 or 4.5 in the text)

Non-homogeneous SL equation:

$$\begin{aligned} -(p(x)u')' + q(x)u &= f(x), \qquad a < x < b \\ u(a) = u(b) &= 0 \qquad \text{(any other self-adjoint BC will also work)} \end{aligned}$$

Given f, we want to solve for u.

$$\begin{cases} Lu = f \\ B(u) = 0 \end{cases} \qquad u = L^{-1} f$$

Does an inverse exist?
If 0 is <u>not</u> an eigenvalue of L, then L is one-to-one and an inverse exists.

Assume L is one-to-one $\Leftrightarrow \lambda = 0$ is not an eigenvalue.

Key result:

$$u(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi$$

where $G(x, \xi)$ is the *Green's function*. In other words, the inverse of a (linear) differential operator is an integral operator with kernel $G(x, \xi)$.

$$\begin{cases} Lg = \delta(x - \xi) \\ B(G) = 0 \end{cases}$$

$$f(x) = \int_a^b \delta(x - \xi) f(\xi)\, d\xi$$

9 1-30-12

9.1 The "δ" Function

Formally, the δ-function satisfies

$$\delta(x) = 0, \qquad x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Thus, δ represents density of a point source at $x = 0$.

We can regard $\delta(x)$ as a limit of functions supported near 0 with integral 1, e.g.

$$f_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & |x| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Can interpret δ as a distribution.

If $f(x)$ is a function that is continuous at 0, then

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0)$$

Note: we don't need to integrate from $-\infty$ to ∞, we simply need to integrate over the support of the δ function.

In particular,

$$\int_{-\infty}^{x} \delta(t)\,dt = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} = H(x) \qquad \text{(step function)}$$
$$H(x) = \int_{-\infty}^{x} \delta(t)\,dt$$
$$\frac{dH}{dt} = \delta(x).$$

More generally, we can take the δ-function supported at ξ: $\delta(x - \xi)$. This has the properties

$$\delta(x-\xi) = 0, \qquad x \neq \xi$$
$$\int_{-\infty}^{\infty} \delta(x-\xi)\,dx = 1$$
$$\underbrace{\int_{-\infty}^{\infty} \delta(x-\xi) f(x)\,dx}_{=\delta * f} = f(\xi)$$
$$\frac{d}{dx} H(x-\xi) = \delta(x-\xi)$$

9.2 Green's Functions

Consider a Sturm-Liouville problem (or other linear differential equation):

$$Lu = f \tag{9.1}$$
$$B(u) = 0.$$

e.g.

$$L = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$
$$B(u): \qquad u(a) = u(b) = 0$$

Then the Green's function, $G(x,\xi)$, is the solution of

$$LG = \delta(x-\xi)$$
$$B(G) = 0$$

The solution of (9.1) can be represented as

$$u(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi$$

To see this:

$$f(x) = \int_a^b f(\xi)\delta(x-\xi)\,dx$$

Linearity is crucial because we are superpositioning solutions at each point. Alternatively,

$$\begin{aligned} Lu(x) &= L\int_a^b G(x,\xi)f(\xi)\,d\xi \\ &= \int_a^b LG(x,\xi)f(\xi)\,d\xi \\ &= \int_a^b \delta(x-\xi)f(\xi)\,d\xi \\ &= f(x). \end{aligned}$$

$$\begin{aligned} u &= L^{-1}f \\ u &= Gf \\ Gf(x) &= \int_a^b G(x,\xi)f(\xi)\,d\xi \end{aligned}$$

Thus, the inverse of the differential L operator is an integral operator with kernel G.

Example 9.1.

Consider

$$-u'' = f(x), \qquad 0 < x < 1 \tag{9.2}$$
$$u(0) = u(1) = 0$$

(This is the SLP with $L = -\frac{d^2}{dx^2}$ and Dirichlet BC's. For example, this could be a model for steady temperature distribution in a rod with sources $f(x)$. The heat equation would be $u_t = u_{xx} + f(x)$, and the steady state is given by (9.2). Or it could be the steady state of a wave equation, $u_{tt} = u_{xx} + f(x)$, where f is the force density.)

Find the Green's function $G(x, \xi)$ for this problem, which satisfies

$$-\frac{d^2}{dx^2}G(x,\xi) = \delta(x-\xi)$$
$$G(0,\xi) = 0$$
$$G(1,\xi) = 0$$

So we need:

$$-\frac{d^2G(x,\xi)}{dx^2} = 0, \qquad x \neq \xi$$
$$G(0,\xi) = 0$$
$$G(1,\xi) = 0$$
$$\left[-\frac{dG}{dx}\right]_\xi = -\frac{dG}{dx}(\xi^+,\xi) + \frac{dG}{dx}(\xi^-,\xi)$$

If $0 \leq x < \xi$, then we need

$$\frac{d^2G}{dx^2} = 0 \quad \Rightarrow \quad G(x,\xi) = \cancel{c_1(\xi)} + c_2(\xi)x$$
$$G(0,\xi) = 0 \quad \Rightarrow \quad G(x,\xi) = c(\xi)x, \qquad 0 \leq x < \xi.$$

If $\xi < x \leq 1$, then we need

$$\frac{d^2G}{dx^2} = 0$$
$$G(1,\xi) = 0 \quad \Rightarrow \quad G(x,\xi) = d(\xi)(1-x).$$

And for the jump:

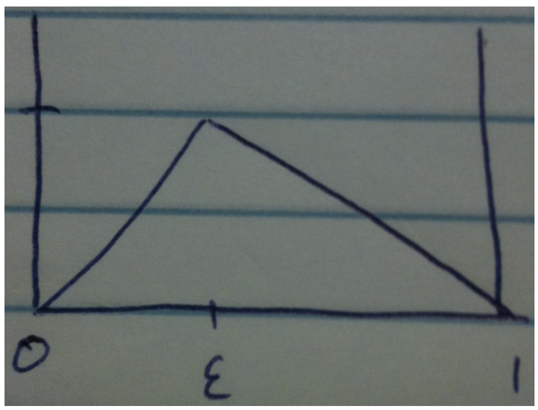
$$\left[-\frac{dG}{dx}\right]_{x=\xi} = -\left.\frac{dG}{dx}\right|_{x=\xi^+} + \left.\frac{dG}{dx}\right|_{x=\xi^-} = d + c = 1$$

Example 9.2. ***Continued...***

G is continuous at ξ, so

$$\begin{aligned} c\xi &= d(1-\xi) \\ d &= 1-c \\ c\xi &= 1-\xi-c(1-\xi) \\ \cancel{c\xi}+c-\cancel{c\xi} &= 1-\xi \\ d &= \xi \end{aligned}$$

$$G(x,\xi) = \begin{cases} (1-\xi)x & 0 \le x < \xi \\ \xi(1-x) & \xi < x \le 1 \end{cases}$$



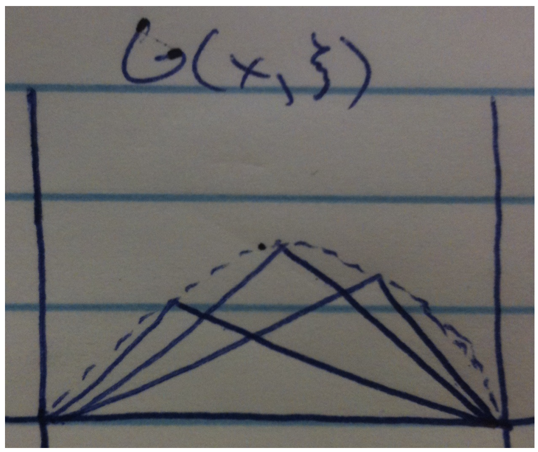
10 2-1-12

10.1 Green's Functions

$$-u'' = f(x), \qquad 0 < x < 1$$
$$u(0) = u(1) = 0$$

Green's function $G(x, \xi)$

$$-\frac{d^2}{dx^2}G(x,\xi) = \delta(x-\xi), \qquad 0 < x < 1$$
$$G(0,\xi) = G(1,\xi) = 0$$
$$G(x,\xi) = \begin{cases} (1-\xi)x & 0 \le x < \xi \\ \xi(1-x) & \xi < x \le 1 \end{cases}$$



Alternatively, we can write

$$G(x,\xi) = x_<(1-x_>),$$

where $x_< = \min(x,\xi)$ and $x_> = \max(x,\xi)$.
G is symmetric:

$$G(x,\xi) = G(\xi,x).$$

Reciprocity: the response at x due to a source at ξ = the response at ξ due to a source at x. (This symmetry is a consequence of self-adjointness.)

$$u(x) = \int_0^1 G(x,\xi)f(\xi)\,d\xi$$

Note:

1. $$u(0) = \int_0^1 G(0,\xi)f(\xi)\,d\xi, \quad u(1) = 0$$

2. Formally,

$$\begin{aligned} -u''(x) &= -\frac{d^2}{dx^2}u = -\frac{d^2}{dx^2}\int_0^1 G(x,\xi)f(\xi)\,d\xi \\ &= \int_0^1 \left[-\frac{d^2}{dx^2}G(x,\xi)\right] f(\xi)\,d\xi \\ &= \int_0^1 \delta(x-\xi)f(\xi)\,d\xi \\ &= f(x) \end{aligned}$$

(Note: The Green's function depends on the boundary conditions.)

Explicitly,

$$
\begin{aligned}
u(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi = (1-x)\int_0^x \xi f(\xi)\, d\xi + x\int_x^1 (1-\xi) f(\xi)\, d\xi \\
u'(x) &= -\int_0^x \xi f(\xi)\, d\xi + \cancel{(1-x)xf(x)} + \int_x^1 (1-\xi) f(\xi)\, d\xi - \cancel{x(1-x)f(x)} \\
u''(x) &= -xf(x) - (1-x)f(x) = -f(x).
\end{aligned}
$$

Also, it is easy to see that $u(0) = u(1) = 0$.

10.2 General SL Problem (Regular)

$$
\begin{aligned}
-(pu')' + qu = f(x), &\qquad a < x < b \\
u(a) = u(b) = 0 &
\end{aligned}
$$

The interval is finite, $p(x), p'(x), q(x)$ are all continuous on $[a,b]$, $p(x) > 0$ on $[a,b]$. We consider Dirichlet boundary conditions, but any self-adjoint boundary conditions will work the same way.

Green's function $G(x,\xi)$:

$$
\begin{aligned}
LG = -\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) + q(x)G = \delta(x-\xi), &\qquad a < x < b \\
G(a,\xi) = G(b,\xi) = 0 & \\
L = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q &
\end{aligned}
$$

We want

$$
\begin{aligned}
LG(x,\xi) &= 0, &\quad & a \le x < \xi \quad \text{with } G(a,\xi) = 0 \\
LG(x,\xi) &= 0, && \xi < x \le b \quad \text{with } G(b,\xi) = 0 \\
[G]_{x=\xi} &= 0, && \text{where } [f]_{x=\xi} = \underbrace{f(\xi^+)}_{\lim_{x\to\xi^+} f(x)} - \underbrace{f(\xi^-)}_{\lim_{x\to\xi^-} f(x)}
\end{aligned}
$$

$$
\left[-p\frac{dG}{dx}\right]_{x=\xi} = 1 \quad \Leftrightarrow \quad \left[\frac{dG}{dx}\right]_{x=\xi} = -\frac{1}{p(\xi)}
$$

Let $u_1(x)$ be the solution of the homogeneous equation with BC at $x = a$:

$$
-(pu_1')' + qu_1 = 0, \qquad u_1(a) = 0.
$$

Let $u_2(x)$ be the solution of the homogeneous equation with BC at $x = b$:

$$
-(pu_2')' + qu_2 = 0, \qquad u_2(b) = 0.
$$

(We know these exist from ODE theory.) If u_1 and u_2 are not independent, then 0 is an eigenvalue and thus we may not have a unique solution. Therefore, we assume the only solution of the homogeneous problem $Lu = 0$, $u(a) = u(b) = 0$, is the zero solution, i.e. $\lambda = 0$ is not an eigenvalue. Then u_1, u_2 are linearly independent. i.e. the Wronskian,

$$
\begin{aligned}
W(u_1,u_2) &= u_1u_2' - u_1'u_2 \\
&= \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}
\end{aligned}
$$

is not identically zero.

$$\begin{aligned}
\frac{d}{dx}(pW) &= \frac{d}{dx}(u_1 \cdot pu_2' - u_2 \cdot pu_1') \\
&= u_1(pu_2')' + \cancel{u_1' \cdot pu_2'} - u_2(pu_1')' - \cancel{u_2' \cdot pu_1'} \\
&= u_1 \cdot qu_2 - u_2 \cdot qu_1 \\
&= 0 \\
p(u_1u_2' - u_1'u_2) &= \text{ constant}
\end{aligned}$$

1.

$$G(x,\xi) = \begin{cases} A(\xi)u_1(x) & a \le x < \xi \\ B(\xi)u_2(x) & \xi < x \le b \end{cases}$$

2. $[G]_{x=\xi} = 0$

$$G(x,\xi) = \begin{cases} cu_2(\xi)u_1(x) & a \le x < \xi \\ cu_1(\xi)u_2(x) & \xi < x \le b \end{cases}$$

3.

$$\begin{aligned}
\left[-p\frac{dG}{dx}\right]_{x=\xi} &= 1 \\
-pc\left[u_1u_2' - u_1'u_2\right]_{x=\xi} &= 1 \\
c &= -\frac{1}{pW(u_1,u_2)} \quad \leftarrow \text{ constant, nonzero}
\end{aligned}$$

11 2-3-12

11.1 Green's Functions

Regular SLP:

$$Lu = f, \qquad L = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x), \quad a < x < b$$
$$B(u) = \begin{pmatrix} u(a) \\ u(b) \end{pmatrix} = 0$$

The Green's function:

$$\begin{aligned} LG &= \delta(x-\xi) \\ B(G) &= 0 \\ G(x,\xi) &= \text{ Green's function} \end{aligned}$$

Integral representation of the solution to the original problem:

$$u(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$$

From last time:

$$G(x,\xi) = \begin{cases} \frac{1}{c}u_1(x)u_2(\xi) & a \le x < \xi \\ \frac{1}{c}u_1(\xi)u_2(x) & \xi < x \le b \end{cases}$$

where

$$\begin{aligned} Lu_1 &= 0, \qquad u_1(a) = 0 \\ Lu_2 &= 0, \qquad u_2(b) = 0 \\ c &= -p(u_1u_2' - u_2u_1') \end{aligned}$$

c is constant, provided u_1 and u_2 are linearly independent ($c \neq 0$). $\lambda = 0$ not an eigenvalue of $L \Rightarrow L$ is invertible. u_1 and u_2 are unique up to multiplication by a constant (which goes away when we divide by the Wronskian).

Example 11.1.

$$\begin{aligned} -\frac{du^2}{dx^2} &= f(x), \qquad 0 < x < 1, \quad L = -\frac{d^2}{dx^2} \\ u(0) &= u(1) = 0 \end{aligned}$$

$$\begin{aligned} u_1(x) &= x \\ u_2(x) &= 1 - x \\ c &= -1[x \cdot (-1) - (1-x) \cdot 1] \\ &= 1 \\ G(x,\xi) &= \begin{cases} x(1-\xi) & 0 \le x \le \xi \\ \xi(1-x) & = \xi \le x \le 1 \end{cases} \end{aligned}$$

11.2 Connection with Spectral Theory

$$\begin{aligned} Lu &= \lambda u + f(x), \qquad a < x < b, \quad \lambda \in \mathbb{C} \\ B(u) &= 0 \end{aligned}$$

If λ is not an eigenvalue of L, then we have a Green's function $G(x, \xi; \lambda)$. (Repeat what we did before with q replaced by $q - \lambda$.) The unique solution is given by

$$\begin{aligned} u(x; \lambda) &= \int_a^b G(x, \xi; \lambda) f(\xi)\, d\xi \\ (L - \lambda)u &= f \\ u &= (L - \lambda)^{-1} f \\ &= R(\lambda) f, \qquad \text{where } R(\lambda) = (L - \lambda)^{-1} \text{ is the } \textit{resolvent} \text{ of } L \\ R(\lambda) f(x) &= \int_a^b G(x, \xi; \lambda) f(\xi)\, d\xi \end{aligned}$$

Suppose that we look for eigenfunctions ϕ of L with eigenvalue λ:

$$\begin{aligned} L\phi &= \lambda \phi \\ B(\phi) &= 0 \end{aligned}$$

$$\begin{aligned} L\phi - \gamma\phi &= (\lambda - \gamma)\phi, \qquad \gamma \in \mathbb{C} \text{ is not an eigenvalue of } L \\ (L - \gamma I)\phi &= (\lambda - \gamma)\phi, \qquad B(\phi) = 0 \\ \Rightarrow \qquad \phi &= (\lambda - \gamma) R(\gamma)\phi \\ \Rightarrow \qquad R(\gamma)\phi &= \mu\phi, \qquad \mu = \frac{1}{\lambda - \gamma} \\ \int_a^b G(x, \xi; \lambda)\phi(\xi)\, d\xi &= \mu\phi(x) \end{aligned}$$

μ expresses eigenvalue of L in terms of eigenvalues of R. $R(\gamma)$ is a compact operator on $L^2(a, b)$ and it is self-adjoint for $\gamma \in \mathbb{R}$ ($G(x, \xi; \gamma) = G(\xi, x; \gamma)$). The general theory of compact self-adjoint operators on Hilbert spaces implies that $R(\gamma)$ has a complete orthonormal set of eigenfunctions (with real eigenvalues), so L has them also. (The key here is that the resolvent is compact.)

11.3 Eigenfunction Expansions

$$\begin{aligned} Lu &= \lambda u + f(x) \\ B(u) &= 0 \end{aligned}$$

Assume that λ is not an eigenvalue of L. Denote the eigenvalues by λ_n:

$$\begin{aligned} L\phi_n &= \lambda_n \phi_n, \qquad n = 1, 2, 3, \ldots \\ B(\phi_n) &= 0 \\ (\phi_m, \phi_n) &= \int_a^b \phi_m \overline{\phi_n}\, dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \end{aligned}$$

Expand u and f as

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x), \qquad c_n = (u, \phi_n) \\ f(x) &= \sum_{n=1}^{\infty} f_n \phi_n(x), \qquad f_n = (f, \phi_n) = \int_a^b f(\xi)\overline{\phi_n(\xi)}\, d\xi \end{aligned}$$

Then

$$
\begin{aligned}
(L-\lambda I)u &= (L-\lambda I)\left(\sum_{n=1}^{\infty} c_n\phi_n\right) \\
&= \sum c_n(L-\lambda I)\phi_n)\phi_n \\
&= \sum(\lambda_n-\lambda)c_n\phi_n, \qquad (L-\lambda I)u=f \\
\sum(\lambda_n-\lambda)c_n\phi_n &= \sum f_n\phi_n \\
(\lambda_n-\lambda)c_n &= f_n \\
c_n &= \frac{f_n}{\lambda_n-\lambda}
\end{aligned}
$$

So the solution is

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty}\frac{f_n}{\lambda_n-\lambda}\phi_n(x) \\
&= \sum_{n=1}^{\infty}\frac{1}{\lambda_n-\lambda}\left[\int_a^b f(\xi)\overline{\phi_n(\xi)}\,d\xi\right]\phi_n(x) \\
&= \int_a^b\left[\sum_{n=1}^{\infty}\frac{\phi_n(x)\overline{\phi_n(\xi)}}{\lambda_n-\lambda}\right]f(\xi)\,d\xi \\
&= \int_a^b G(x,\xi;\lambda)f(\xi)\,d\xi
\end{aligned}
$$

Thus, we have the *bilinear formula* for the Green's function:

$$
G(x,\xi;\lambda) = \sum_{n=1}^{\infty}\frac{\phi_n(x)\overline{\phi_n(\xi)}}{\lambda_n-\lambda}
$$

12 2-6-12

12.1 Completeness Property of δ

Suppose that $\{\phi_1, \phi_2, \phi_3, \dots\}$ is a complete orthonormal set in $L^2(a, b)$.

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\overline{\phi_n(x)}\, dx = \delta_{mn}$$

For some $a < \xi < b$, expand $\delta(x - \xi)$ w.r.t. $\{\phi_n\}$:

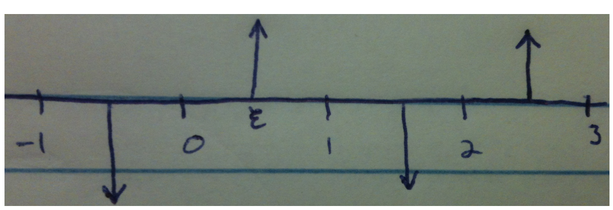
$$\begin{aligned}
\delta(x-\xi) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\
c_n &= \int_a^b \delta(x-\xi)\overline{\phi_n(x)}\, dx = \overline{\phi_n(\xi)} \\
\delta(x-\xi) &= \sum_{n=1}^{\infty} \phi_n(x)\overline{\phi_n(\xi)}
\end{aligned}$$

Conversely, suppose $f \in L^2(a, b)$.

$$\begin{aligned}
f(x) &= \int_a^b \delta(x-\xi) f(\xi)\, d\xi \\
&= \int_a^b \sum_{n=1}^{\infty} \phi_n(x)\overline{\phi_n(\xi)} f(\xi)\, d\xi \\
&= \sum_{n=1}^{\infty} f_n \phi_n(x) \\
f_n &= \int_a^b f(\xi)\overline{\phi_n(\xi)}\, d\xi = (f, \phi_n)
\end{aligned}$$

Example 12.1.

$$\begin{aligned}
\phi_n &= \sqrt{2}\sin(n\pi x) \qquad \text{in } L^2(0,1),\ n = 1, 2, 3, \dots \\
\delta(x-\xi) &= \sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi\xi) \qquad 0 < x, \xi < 1
\end{aligned}$$



12.2 Eigenfunction Expansions

$$Lu = \lambda u + f(x), \qquad a < x < b, \quad L = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x),\ \lambda \in \mathbb{C} \text{ (not an eigenvalue of } L)$$

$$B(u) = 0 = \begin{pmatrix} u(a) \\ u(b) \end{pmatrix}$$

Assume to be a regular SL problem. We have an orthonormal basis of eigenfunctions $\{\phi_1, \phi_2, \phi_3, \ldots\}$ with real eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$, $\lambda_1 < \lambda_2 < \lambda_3 < \ldots,\ \lambda_n \to \infty$.

$$u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)$$

Diagonalize the equation:

$$\begin{aligned}
Lu(x) &= \sum_{n=1}^{\infty} \lambda_n c_n \phi_n(x) \\
(L - \lambda I)u &= \sum_{n=1}^{\infty} (\lambda_n - \lambda) c_n \phi_n(x) \\
&= \sum_{n=1}^{\infty} f_n \phi_n(x) \\
(\lambda_n - \lambda) c_n &= f_n \\
c_n &= \frac{f_n}{\lambda_n - \lambda}, \qquad \lambda \neq \lambda_n \\
u(x) &= \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} \phi_n(x) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n - \lambda} \right) \left[\int_a^b f(\xi) \overline{\phi_n(\xi)}\, d\xi \right] \phi_n(x) \\
&= \int_a^b G(x, \xi; \lambda) f(\xi)\, d\xi \\
G(x, \xi; \lambda) &= \sum_{n=1}^{\infty} \frac{\phi_n(x) \overline{\phi_n(\xi)}}{\lambda_n - \lambda}
\end{aligned}$$

$$\begin{aligned}
(L - \lambda I) G(x, \xi; \lambda) &= \sum_{n=1}^{\infty} \frac{\overline{\phi_n(\xi)}\, \overbrace{(L - \lambda I)\phi_n(x)}^{(\lambda_n - \lambda)\phi_n}}{\lambda_n - \lambda} \\
&= \sum_{n=1}^{\infty} \phi_n(x) \overline{\phi_n(\xi)} \\
&= \delta(x - \xi)
\end{aligned}$$

Example 12.2.

$$-u'' = \lambda u + f(x), \qquad 0 < x < 1, \quad L = -\frac{d}{dx^2}$$
$$u(0) = u(1) = 0$$

Eigenfunctions & Eigenvalues:

$$-\phi_n'' = \lambda_n \phi_n$$
$$\lambda_n(0) = \lambda_n(1) = 0$$
$$\phi_n(x) = \sqrt{2}\sin(n\pi x)$$
$$\lambda_n = n^2\pi^2, \qquad n = 1, 2, 3, \ldots$$

The Green's function will satsify

$$-\frac{d^2G}{dx^2} = \lambda G + \delta(x-\xi)$$
$$G(0,\xi;\lambda) = G(1,\xi;\lambda) = 0$$

Eigenfunction expansion:

$$G(x,\xi;\lambda) = \sqrt{2}\sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi\xi)}{n^2\pi^2 - \lambda}$$

Note: Poles at $\lambda = \lambda_n$.
The series converges uniformly (by M-test).

12.2.1 Comparison with the Explicit Solution

$$-\frac{d^2G}{dx^2} = \lambda G + \delta(x-\xi)$$
$$G(0,\xi;\lambda) = G(1,\xi;\lambda) = 0$$
$$G(x,\xi;\lambda) = 2\sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi\xi)}{n^2\pi^2 - \lambda}$$

$$G(x,\xi;\lambda) = \begin{cases} \frac{1}{c}u_1(x;\lambda)u_2(\xi,\lambda) & 0 \le x < \xi \\ \frac{1}{c}u_1(\xi;\lambda)u_2(x,\lambda) & \xi < x \le 1 \end{cases}$$
$$-u_1'' = \lambda u_1, \quad u_1(0;\lambda) = 0$$
$$-u_2'' = \lambda u_2, \quad u_2(1;\lambda) = 0$$
$$c = -(u_1u_2' - u_2u_1'), \qquad (p = 1)$$

Assume $\lambda = k^2 > 0$.

$$\begin{aligned}
-u_1'' &= k^2 u_1, \quad u_1(0;\lambda) = 0 \quad \Rightarrow \quad u_1(x) = \sin(kx) \\
-u_2'' &= k^2 u_2, \quad u_2(1;\lambda) = 0 \quad \Rightarrow \quad u_2(x) = \sin[k(1-x)] \\
u_1 u_2' - u_2 u_1' &= -k\sin(kx)\cos[k(1-x)] - k\sin[k(1-x)]\cos kx \\
&= -k\sin[kx + k(1-x)] \\
&= -k\sin k \qquad \text{(constant)} \\
c &= k\sin k \\
G(x,\xi;\lambda) &= \begin{cases} \frac{\sin(kx)\sin[k(1-\xi)]}{k\sin k} & 0 \le x < \xi \\ \frac{\sin(k\xi)\sin[k(1-x)]}{k\sin k} & \xi < x \le 1 \end{cases}
\end{aligned}$$

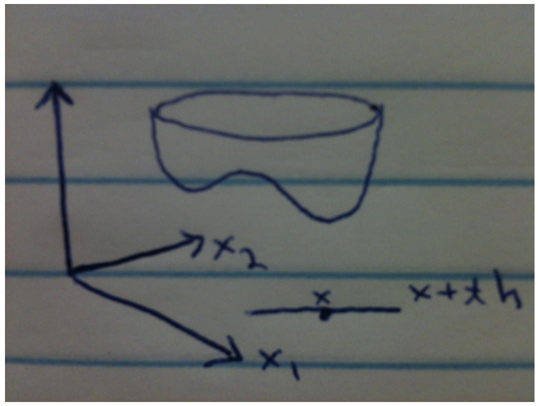
If $\lambda = -k^2$, change $\sin k(\ )$ to $\sinh k(\ )$.

Note that G has poles at $k = n\pi \Leftrightarrow \lambda = n^2\pi^2$ ($\leftarrow$ eigenvalues).

13 2-8-12

13.1 Variational Principles

Consider the finite-dimensional case: $F : \mathbb{R}^n \to \mathbb{R}$ (differentiable). Suppose F has a minimum at $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$. Then x is a critical point of F. Look at the directional derivative of F at x in direction $h \in \mathbb{R}^n$.



$$\begin{aligned}\frac{d}{dt}F(x+th)\Big|_{t=0} &= Df(x)(h) \\ &= \nabla F(x) \cdot h \\ &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} h_i\end{aligned}$$

At a minimum (or maximum), this must be 0 at $h \in \mathbb{R}^n$, so $\nabla F(x) = 0$. If F has an extreme value at x, then x is a critical point of F.

We can have critical points that are neither a max nor min $\Rightarrow$ saddle point.

Indirect method: look for critical points that satisfy $\nabla F(x) = 0$, search among those for a minimizer.

Direct method: look for minima of F.

Example 13.1.

$$F(x, y) = x^4 + 25x^2y + x + y^6$$

At a critical point:

$$\begin{aligned}4x^3 + 50xy + 1 &= 0 \\ 25x^2 + 6y^5 &= 0\end{aligned}$$

We know this has a solution because F is continuous and $F(x, y) \to \infty$ as $x, y \to \pm\infty$. So this problem has (at least) one real solution since F attains a minimum.

Suppose we have a system of equations:

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ f_2(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \ldots, x_n) &= 0 \end{aligned}$$

Can we write them as $\nabla F = 0$?

$$f_i = \frac{\partial F}{\partial x_i} \quad \Leftrightarrow \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \left(= \frac{\partial^2 F}{\partial x_i \partial x_j} \right)$$

If we changed the previous example to:

$$\begin{aligned} 4x^3 - 50xy + 1 &= 0 \\ 25x^2 + 6y^5 &= 0 \end{aligned}$$

then we can't use our variational argument.

13.2 Quadratic Variational Principles

$$\begin{aligned} F(x) &= \frac{1}{2} x^T A x - b^T x \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j - \sum_{i=1}^{n} b_i x_i \end{aligned}$$

where A is an $n \times n$ (symmetric) matrix and $b \in \mathbb{R}^n$. Critical points:

$$\nabla F(x) = Ax - b$$

So $Ax = b$ at a critical point ($A^T = A$).

13.3 Sturm-Liouville Problems

$$J(u) = \int_a^b \frac{1}{2} p(x) [u'(x)]^2 + \frac{1}{2} q(x) u^2(x) - f(x) u(x) \, dx$$

defined on a vector space of functions such that $u(a) = u(b) = 0$. Here, $p(x), q(x), f(x)$ are given coefficient functions (smooth). J is called a *(quadratic) functional*.

$$u \in H^1(a, b) = \{u \bigm| u, u' \in L^2(a, b)\}$$

Example 13.2.

$$J(u) = \int_0^1 \frac{1}{2} (u')^2 - x^2 u \, dx \qquad (p = 1, \ q = 0, \ f = x^2)$$

If $u(x) = x(1 - x)$,

$$J(x) = \ldots \quad \text{(a number)}$$

Suppose J attains a minimum at some function $u(x)$. What can we say about u? Let $h(x)$ be any function such that $h(a) = h(b) = 0$.

$$\begin{aligned}
DJ(u)(h) &= \frac{d}{dt} J(u + th)\bigg|_{t=0} \\
&= \frac{d}{dt} \int_a^b \frac{1}{2} p(u' + th')^2 + \frac{1}{2} q(u + th)^2 - f(u + th)\, dx \bigg|_{t=0} \\
&= \frac{d}{dt} \int_a^b \frac{1}{2} p(u'^2 + 2tu'h' + t^2 h'^2) + \frac{1}{2} q(u^2 + 2tuh + t^2 h^2) - fu - tfh\, dx \bigg|_{t=0} \\
DJ(u)(h) &= \int_a^b pu'h' + quh - fh\, dx
\end{aligned}$$

If J attains a minimum at u, then $DJ(u)(h) = 0$ for all h.

Now suppose $u \in C^2[a, b]$. Then we can integrate by parts:

$$DJ(u)(h) = \int_a^b [\underbrace{-(pu')' + qu - f}_{=0}] h\, dx = \int_a^b \left(\frac{\delta J}{\delta u} h \right) dx, \qquad \frac{\delta J}{\delta u} = -(pu')' + qu - f$$

$$-(pu')' + qu = f, \qquad u(a) = u(b) = 0$$

This is the Sturm-Liouville problem.

14 2-10-12

14.1 Variational Principle for SL Problems

$$J(u) = \int_a^b \left(\frac{1}{2}p(u')^2 + \frac{1}{2}qu^2 - fu \right) dx$$

p, p', q, f are continuous, $p(x) > 0$ for $a \le x \le b$. $J : X \to \mathbb{R}$ is a functional on space X of functions u. Natural space on which to define it:

$$X = H_0^1(a,b) = \{u \mid u, u' \in L^2(a,b),\ u(a) = u(b) = 0\}$$

We looked at the directional derivative of J in direction h:

$$\begin{aligned}
\frac{d}{dt}J(u+th)\bigg|_{t=0} &= \int_a^b \left(pu'h' + quh - fh\right) dx, \qquad h \in X \\
&= \int_a^b \underbrace{\left(-(pu')' + qu - f\right)}_{=\frac{\delta J}{\delta u}} h\, dx = 0 \qquad \text{if, e.g. } u \in C^2[a,b] \\
&= \int_a^b \frac{\delta J}{\delta u} h\, dx, \qquad \text{where } \frac{\delta J}{\delta u} \text{ is the variational derivative of } J(u)
\end{aligned}$$

Suppose $J(u)$ attains a minimum at some $u \in C^2[a,b]$. Then u must satisfy

$$\begin{aligned}
\frac{d}{dt}J(u+th)\bigg|_{t=0} &= 0 \qquad \text{for all } h \in X \\
\Rightarrow \qquad -(pu')' + qu &= f
\end{aligned}$$

This is called the *Euler-Lagrange equation* for $J(u)$.

Weak formulation of the ODE:

$$\int_a^b (pu'h' + quh - fh)\, dx = 0 \qquad \text{for all } h \in X$$

14.2 Galerkin Methods

$$\begin{aligned}
J(u) &= \frac{1}{2}a(u,u) - (f,u)\,, \qquad u, v \in X \\
\text{where} \qquad a(u,v) &= \int_a^b (pu'v' + quv), \\
(f,u) &= \int_a^b fu\, dx
\end{aligned}$$

With this notation,

$$\begin{aligned}
\frac{d}{dt}J(u+th)\bigg|_{t=0} &= \frac{d}{dt}\left[\frac{1}{2}a(u+th, u+th) - (f, u+th)\right]\bigg|_{t=0} \\
&= \frac{d}{dt}\left[\frac{1}{2}a(u,u) + ta(u,h) + \frac{1}{2}t^2 a(h,h) - (f,u) - t\,(f,h)\right]\bigg|_{t=0} \\
&= a(u,h) - (f,h)
\end{aligned}$$

So if $u \in X$ minimizes $J(u)$, then $a(u,h) = (f,h)$ for all $h \in X$. This is the *weak form of the Euler-Lagrange equation*.

Remark 14.1. ***Aside...***

Suppose $u \in C^1(a,b)$. Then

$$\int_a^b u' h \, dx = -\int_a^b u h' \, dx, \qquad h(a) = h(b) = 0$$

We define the weak derivative $v = u'$ by

$$\int_a^b u h' \, dx = -\int_a^b v h \, dx \qquad \text{for all } h.$$

Look for a finite dimensional approximation of the solution $u_N \in X_N$, where $X_N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$, $\phi_j \in X$,

$$u_N(x) = \sum_{j=1}^N c_j \phi_j(x)$$

We can require that u_N satisfies the *Galerkin approximation*:

$$\begin{aligned}
& a(u_N, h) = (f,h) \qquad \text{for all } h \in X_N \\
\Rightarrow \quad & a(u_N, \phi_j) = (f, \phi_j), \qquad j = 1, 2, \ldots, N \\
\Rightarrow \quad & a\left(\sum_{k=1}^N c_k \phi_k, \phi_j\right) = (f, \phi_j), \qquad j = 1, 2, \ldots, N \\
\Rightarrow \quad & \sum_{k=1}^N a_{jk} c_k = b_j, \qquad a_{jk} = a(\phi_j, \phi_k), \; b_j = (f, \phi_j) \\
\Rightarrow \quad & \mathbf{A}\mathbf{c} = \mathbf{b}
\end{aligned}$$

This is a matrix equation. Equivalently, we can define

$$J_N(\mathbf{c}) = J\left(\sum_{j=1}^N c_j \phi_j(x)\right)$$

and $u_N \in X_N$ is the solution that minimizes $J_N(\mathbf{c})$.

14.3 Finite Element Method

Uses piecewise polynomial basis functions supported on intervals (triangles, simplices, etc.). $a_{jk} = a(\phi_j, \phi_k)$, $A = [a_{jk}]$ is a tridiagonal matrix.

15 2-13-12

15.1 Variational Principles for Eigenvalues

$$-(pu')' + qu = \lambda u, \qquad a < x < b$$
$$u(a) = u(b) = 0$$

We can write this as $Lu = \lambda u$. We have a sequence of eigenvalues $\lambda_1 < \lambda_2 < \dots$, with eigenfunctions $\phi_1(x), \phi_2(x), \dots$.

Definition 15.1. ***Rayleigh Quotient***

$$R(u) = \frac{\int_a^b [p(u')^2 + qu^2]\,dx}{\int_a^b u^2\,dx} = \frac{a(u,u)}{\|u\|^2}$$

where

$$\|u\|^2 = \int_a^b u^2\,dx$$
$$a(u,v) = \int_a^b [pu'v' + quv]\,dx \overset{\text{IBP}}{=} \int_a^b Lu \cdot v\,dx$$

Suppose

$$u(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \qquad c_n = (u, \phi_n) = \int_a^b u(x)\overline{\phi_n(x)}\,dx, \quad \|\phi_n\| = 1$$
$$a(u,u) = (Lu, u) = \left(\sum_{n=1}^{\infty} \lambda_n c_n \phi_n, \sum_{m=1}^{\infty} c_m \phi_m\right) = \sum_{m,n=1}^{\infty} \lambda_n c_n \overline{c_m}\,(\phi_n, \phi_m) = \sum_{n=1}^{\infty} \lambda_n |c_n|^2$$
$$\|u\|^2 = \sum_{n=1}^{\infty} |c_n|^2$$
$$R(u) = \frac{\sum_{n=1}^{\infty} \lambda_n |c_n|^2}{\sum_{n=1}^{\infty} |c_n|^2}$$

What is $\min\limits_{u \in H_0^1(a,b)} R(u)$? Answer: $\lambda_1 = \min R(u)$.

Alternative point of view: minimize $a(u,u)$ subject to the constraint that $\|u\|^2 = 1$. We introduce a Lagrange multiplier λ and look for critical points of

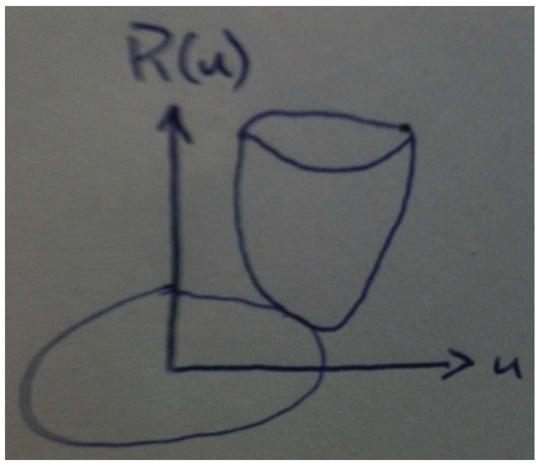
$$I(u,\lambda) = a(u,u) - \lambda(\|u\|^2 - 1)$$

This gives us:

$$\frac{\partial I}{\partial \lambda} = 0 \quad \Rightarrow \quad \|u\|^2 = 1$$
$$\frac{\delta I}{\delta u} = 0 \quad \Rightarrow \quad Lu = \lambda u$$

We can use this principle to get upper bounds/approximations of the smallest eigenvalue of L. If S_k is any k-dimensional subspace of functions (satisfying the BC's),

$$\lambda_1 \le \min_{u \in S_k} R(u)$$



Example 15.2.

$$-u'' = \lambda u, \qquad 0 < x < 1$$
$$u(0) = u(1) = 0$$

$$R(u) = \frac{\int_0^1 (u')^2\,dx}{\int_0^1 u^2\,dx}$$

Trial function:

$$u(x) = x(1-x)$$
$$u'(x) = 1 - 2x$$
$$R(x(1-x)) = \frac{\int_0^1 (1 - 4x + 4x^2)\,dx}{\int_0^1 x^2 - 2x^3 + x^4\,dx}$$
$$= 10 \ge \lambda_1 = \pi^2 \approx 9.87$$

$$
\begin{aligned}
R(u) &= \frac{\int_a^b [p(u')^2 + qu^2]\,dx}{\int_a^b u^2\,dx} \\
p(x) &> 0 \text{ on } [a,b] \\
q(x) &\geq 0 \text{ on } [a,b] \\
\Rightarrow \quad 0 &< \lambda_1
\end{aligned}
$$

All eigenvalues are positive (for Dirichlet BC's). Zero cannot be an eigenvalue because this would imply that $u' = 0$ and $u(0) = u(1) = 0$, which implies that $u = 0$.

We can get min-max variational principles for higher eigenvalues.

$$\lambda_k = \min_{S_k} \left[\max_{u \in S_k} R(u) \right]$$

taken over all k-dimensional subspaces S_k.

15.2 Singular SL Problems

$$-(pu')' + qu = \lambda u, \qquad a < x < b$$

In a regular problem, we have:

1. $[a,b]$ is a finite interval
2. p, p', q are continuous on $[a,b]$
3. $p(x) > 0$ for $x \in [a,b]$

The two common ways that these fail are:

1. have an infinite interval (e.g. $a = -\infty$ and/or $b = \infty$)
2. $p(x) > 0$ for $x \in (a,b)$ but $p(a) = 0$ and/or $p(b) = 0$

Then we get a singular SL problem.

- Endpoint a is singular if $a = -\infty$ or $p(a) = 0$
- Endpoint b is singular if $b = \infty$ or $p(b) = 0$

Example 15.3.

(a)

$$-u'' = \lambda u, \quad -\infty < x < \infty$$

Both endpoints are singular

(b)

$$-u'' = \lambda u, \quad 0 < x < \infty$$

The right endpoint is singular

(c)

$$[(1 - x^2)u']' = \lambda u, \quad -1 < x < 1$$

Both endpoints are singular

(d)

$$-(xu')' = \lambda u, \quad 0 < x < 1$$

The left endpoint is singular

16 2-15-12

16.1 A Singular SLP

$$u'' = \lambda u, \qquad -\infty < x < \infty, \qquad L = -\frac{d^2}{dx^2}$$

Look for solutions with $\lambda \in \mathbb{C}$.

$$\begin{aligned} u(x) &= e^{kx} \\ -k^2 &= \lambda \\ k &= \pm\sqrt{-\lambda} \\ \textbf{Choose } \operatorname{Re}\sqrt{-\lambda} &> 0 \end{aligned}$$

$-\lambda$ is <u>not</u> a nonnegative real number. Note that the square root is discontinuous; we call the negative part of the real axis the *branch cut*.

Consider the case when λ is not on the positive real axis. The general solution of the ODE is

$$u(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

To avoid an unbounded solution, we need $c_1 = c_2 = 0 \iff u = 0$. Thus, λ is <u>not</u> in the spectrum of L.

Consider the case when $0 \le \lambda < \infty$. Then

$$\begin{aligned} \pm\sqrt{-\lambda} &= \pm ik, \qquad \text{where } k^2 = \lambda,\ 0 \le k < \infty \\ u(x) &= c_1 e^{ikx} + c_2 e^{-ikx}. \end{aligned}$$

This is a bounded function of x. All real $\lambda \ge 0$ are in the spectrum of L (continuous spectrum). No eigenfunctions $u \in L^2(\mathbb{R})$.

Regular SLP:

$$\begin{aligned} -u'' &= \lambda u, \qquad 0 < x < 1 \\ u(0) &= u(1) = 0 \end{aligned}$$

The spectrum is a discrete sequence, $\{\pi^2, 4\pi^2, \ldots, n^2\pi^2, \ldots\}$ that goes off to infinity. This is a point spectrum of eigenvalues.

Singular SLP:

$$-u'' = \lambda u, \qquad 0 < x < \infty$$

The spectrum is $0 \le \lambda < \infty$. This is a continuous spectrum. (But not every singular SLP has a continuous spectrum.)

16.2 Green's Function for a Singular SLP

$$\begin{aligned} -u'' &= \lambda u + f(x), \qquad -\infty < x < \infty, \quad f \in L^2(\mathbb{R}) \\ u &\in L^2(\mathbb{R}) \\ -\frac{d^2G}{dx^2} &= \lambda G + \delta(x-\xi), \qquad G(x,\xi;\lambda) = \text{ Green's function} \\ G &\in L^2(\mathbb{R}) \end{aligned}$$

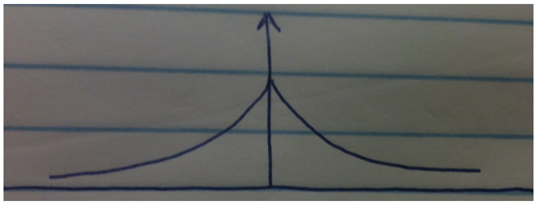
Solutions of the homogeneous equation: $e^{-\sqrt{-\lambda}x}, e^{\sqrt{-\lambda}x}, \ \lambda \in \mathbb{C}, \ \lambda$ is <u>not</u> $0 \le \lambda < \infty$.

$$G(x,\xi;\lambda) = \begin{cases} \frac{e^{-\sqrt{-\lambda}\xi} e^{\sqrt{-\lambda}x}}{2\sqrt{-\lambda}} & -\infty < x < \xi \\ \frac{e^{\sqrt{-\lambda}\xi} e^{-\sqrt{-\lambda}x}}{2\sqrt{-\lambda}} & \xi < x < \infty \end{cases}$$

Example 16.1.

If $\lambda = -1$,

$$G(x,\xi;-1) = \begin{cases} \frac{1}{2} e^{-\xi} e^{x} & -\infty < x < \xi \\ \frac{1}{2} e^{\xi} e^{-x} & \xi < x < \infty \end{cases} \qquad = \frac{1}{2} e^{-|x-\xi|}$$



Solution:

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi;\lambda) f(\xi)\, d\xi$$
$$u = (L - \lambda I)^{-1} f$$

In the regular SLP case, we saw that $G(x,\xi;\lambda)$ has poles at the eigenvalues. In the singular SLP case, we can define $G(x,\xi;\lambda)$ everywhere in the complex plane *except* at the branch cut.

16.2.1 Fourier Transform

Instead of an eigenfunction expansion (associated with the point spectrum of eigenvalues), we get an integral transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk, \qquad f \in L^2(\mathbb{R})$$
$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

(Think of this integral as a sum and compare to the regular case.)

16.2.2 δ-function and Fourier Transforms

$$\hat{\delta}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx = \frac{1}{2\pi}$$
$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk$$

Intuition for $\delta(x)$: sin is odd, so the imaginary part will cancel out. The cos terms will cancel out everywhere except at 0.

17 2-17-12

17.1 Singular Sturm-Liouville Problems

$$-(pu')' + qu = \lambda ru, \qquad a < x < b \tag{17.1}$$

Assume:

- p, p', q, r are continuous in the open interval (a, b)
- $p(x)$ and $r(x)$ are strictly positive on (a, b)

This is a regular SLP if

1. $[a, b]$ is a finite interval
2. p, p', q, r are continuous on $[a, b]$
3. $p(x) > 0$ for $x \in [a, b]$

Otherwise we have a singular SLP. The problem is singular at a if

1. $a = -\infty$
2. $p(a) = 0$
3. (possibly) q, r are unbounded at a

and similarly for b. It is OK for $r(x) = 0$ for some $x \in [a, b]$.

In the regular case with separated, self-adjoint BC's, the spectrum is purely a point spectrum (eigenvalues), with

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots, \quad \lambda_n \to \infty,$$

with a complete set of orthogonal eigenfunctions

$$\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$$

in the space $L^2_r(a, b)$:

$$(\phi_n, \phi_m) = \int_a^b r(x)\phi_n(x)\overline{\phi_m(x)}\, dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

and $u \in L^2_r(a, b)$ if $\int_a^b r(x)|u(x)|^2\, dx < \infty$.

Theorem 17.1. ***Weyl (1910)***

Suppose the SLP is regular at a (a is finite, $p(a) > 0$) and singular at b ($b = \infty$ or $p(b) = 0$). There are two cases:

1. **Limit Circle (LC):** All solutions of (17.1) belong to $L^2_r(a, b)$. This holds for all $\lambda \in \mathbb{C}$ if it holds for any particular $\lambda \in \mathbb{C}$.
2. **Limit Point (LP):** Some solutions of (17.1) that do not belong to $L^2_r(a, b)$.
   - If $\lambda \in \mathbb{C}$ and λ is not real, then exactly one solution belongs to $L^2_r(a, b)$ (up to constant multiples) and other solutions don't. If $\lambda \in \mathbb{R}$, we have at least one solution not in $L^2_r(a, b)$–we may have no solutions in $L^2_r(a, b)$ (except $u = 0$).

If both a, b are singular endpoints, choose $c \in (a, b)$ and classify a in terms of $L_r^2(a, c)$ and b in terms of $L_r^2(c, b)$ (the particular choice of c doesn't matter).

Example 17.2.

Consider $L = -\frac{d^2}{dx^2}$ on three intervals:

(a) $-u'' = \lambda u, \quad 0 < x < \infty$, 0 is regular, ∞ is singular

(b) $-u'' = \lambda u, \quad -\infty < x < 0$, $-\infty$ is singular, 0 is regular

(c) $-u'' = \lambda u, \quad -\infty < x < \infty$, both endpoints are singular

LC or LP?

(a) Consider $\lambda = 0$: $-u'' = 0 \quad \Rightarrow$

$$\begin{aligned} u(x) &= c_1 \cdot 1 + c_2 \cdot x \\ &= c_1 u_1(x) + c_2 u_2(x), \quad u_1(x) = 1,\ u_2(x) = x \end{aligned}$$

Are $u_1, u_2 \in L^2(0, \infty)$? i.e., is

$$\int_0^\infty |u_1|^2 \, dx < \infty, \qquad \int_0^\infty |u_2|^2 \, dx < \infty$$

No. Neither solution is in $L^2(0, \infty) \Rightarrow x = \infty$ is in the LP case.

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$u(x) = c_1 \underbrace{e^{-\sqrt{-\lambda}x}}_{\in L^2(0,\infty)} + c_2 \underbrace{e^{\sqrt{-\lambda}x}}_{\notin L^2(0,\infty)}$$

(b) Same story $\Rightarrow$ LP. $u_1 = 1,\ u_2 = x$.

(c) Both endpoints are LP. Divide the interval at 0 and apply the previous results.

Example 17.3. ***Bessel's equation of order ν***

$$-(xu')' + \frac{\nu^2}{x}u = \lambda xu, \qquad 0 < x < 1,\ \nu \geq 0 \text{ is a real paramter}$$

$$p(x) = x, \quad q(x) = \frac{\nu^2}{x}, \quad r(x) = x$$

0 is singular because p vanishes there. 1 is regular.

If $\lambda = 0$:

$$\begin{aligned} 0 &= -(xu')' + \frac{\nu^2}{x}u \\ &= -xu'' - u' + \frac{\nu^2}{x}u \\ &= -u'' - \frac{1}{x}u' + \frac{\nu^2}{x^2}u \\ &= -x^2u'' - xu' + \nu^2 u \end{aligned}$$

Look for solutions $u(x) = x^r$:

$$\begin{aligned} 0 &= -(rxx^{r-1}) + \nu^2 x^{r-1} \\ &= -r^2 x^{r-1} + \nu^2 x^{r-1} \\ r^2 &= \nu^2, \quad r = \pm\nu \end{aligned}$$

The solution is

$$u(x) = c_1 x^{\nu} + c_2 x^{-\nu}$$

Is

$$\int_0^1 x|u|^2\,dx < \infty \;\Leftrightarrow\; u \in L_x^2(0,1)$$

$$\int_0^1 x \cdot x^{-2\nu}\,dx < \infty$$

$$\int_0^1 \frac{1}{x^{2\nu-1}}\,dx < \infty$$

$$2\nu - 1 < 1, \quad \nu < 1$$

$0 \leq \nu < 1$: LC
$\nu \geq 1$: LP

18 2-22-12

Office Hours: 3-4 today

18.1 Singular Sturm-Liouville Problems

$$-(pu')' + qu = \lambda ru + f(x), \qquad a < x < b$$

with some boundary conditions. Suppose a is a regular endpoint and b is a singular endpoint ($b = \infty$ or $p(b) = 0$).

$$\begin{aligned} L &= \frac{1}{r}\left[-\frac{d}{dx}p\frac{d}{dx} + q\right], \\ Lu &= \lambda u \end{aligned}$$

Introduce a weighted inner product:

$$\begin{aligned} \langle u, v\rangle_r &= \int_a^b r(x)u(x)\overline{v(x)}\, dx \\ \|u\|_r &= \sqrt{\int_a^b r(x)|u(x)|^2\, dx} \end{aligned}$$

$u \in L_r^2(a, b)$ if $\|u\|_r < \infty$.

$$\begin{aligned} \int_a^b r(x)\left[uL\overline{v} - Lu\overline{v}\right]\, dx &= \langle u, Lv\rangle_r - \langle Lu, v\rangle_r \\ &= \int_a^b u\left[\left\{-(p\overline{v}')' + q\overline{v}\right\} - \left\{-(pu')' + qu\right\}\overline{v}\right]\, dx \\ &= \int_a^b \left\{-u(p\overline{v}')' + (pu')'\overline{v}\right\}\, dx \\ &= \int_a^b [pu'\overline{v} - pu\overline{v}']'\, dx \qquad \text{Note: } fg'' - f''g = (fg' - f'g)' \\ &= \left[p(u'\overline{v} - u\overline{v}'\right]_a^b \\ \langle u, Lv\rangle_r - \langle Lu, v\rangle_r &= \int_a^b r\left\{u\overline{Lv} - Lu\overline{v}\right\}\, dx \\ &= [u, \overline{v}](b) - [\overline{v}, u](a), \\ \text{where} \quad [u, \overline{v}] &= p(u'\overline{v} - u\overline{v}') \\ \text{and} \quad L &= \frac{1}{r}\left[-\frac{d}{dx}p\frac{d}{dx} + q\right] \end{aligned}$$

Definition 18.1. ***Admissible***

A function u is *admissible* if $u \in L^2_r(a,b)$ and $Lu \in L^2_r(a,b)$. A complex (or real) number $\lambda \in \mathbb{C}$ is in the resolvent set of L if the equation

$$(L - \lambda I)u = f \qquad + \text{ boundary conditions}$$

has an admissible solution u (unique) for every $f \in L^2_r(a,b)$. Otherwise, we say that λ is in the spectrum of L.

We denote the resolvent set by $\rho(L)$ and the spectrum by $\sigma(L)$.

Comments:

1. If $\lambda \in \rho(L)$ and $(L - \lambda I)u = f$, then

$$u(x) = \int_a^b G(x,\xi;\lambda) f(\xi)\, d\xi$$

where $G(x,\xi;\lambda)$ is the Green's function of $(L - \lambda I)$.

2. If λ is an eigenvalue of L–meaning that there exists $u \in L^2_r(a,b)$, $u \neq 0$, such that $Lu = \lambda u$–then λ is in the spectrum of L.
   - For a regular SLP, the spectrum consists entirely of eigenvalues.

18.2 Weyl Alternative

Consider a SLP on $a < x < b$ that is regular at a and singular at b. We have one of two possibilities:

1. **Limit Circle (LC).** Every solution of the homogeneous equation $Lu = \lambda u$ belongs to $L^2_r(a,b)$. If this is true for one value of λ, then it is true for all $\lambda \in \mathbb{C}$.
2. **Limit Point (LP).** Some solutions are not in $L^2_r(a,b)$.

18.2.1 Limit Circle Case

$$\begin{aligned} Lu &= \lambda u + \frac{f}{r}, \qquad a < x < b,\ a \text{ regular},\ b \text{ singular} \\ u(a) &= 0 \end{aligned}$$

We are looking for a solution $u \in L^2_r(a,b)$. We need a boundary condition at b in order to have a unique solution. So we add the boundary condition:

$$[u,w](b) = \lim_{x \to b} [u,w](x)$$

for some admissible function w. We look for the Green's function for $\lambda = 0$ (or $\lambda = \lambda_0$ if 0 is an eigenvalue).

$$G(x,\xi) = \begin{cases} \frac{1}{c} u_1(x) u_2(\xi) & x < \xi \\ \frac{1}{c} u_1(\xi) u_2(x) & x > \xi \end{cases}$$

Since $u_1, u_2 \in L^2_r(a,b)$, it follows that

$$\int_a^b r(x) r(\xi) |G(x,\xi)|^2 \, dx\, d\xi < \infty$$

This kernel is called a *Hilbert-Schmidt kernel*. This implies that the spectrum consists entirely of eigenvalues.

Bottom line: The limit circle case is very similar to the regular case.

18.2.2 Limit Point Case

$$\begin{aligned} Lu &= \lambda u + f, \qquad a < x < b,\ a \text{ regular},\ b \text{ singular} \\ u(a) &= 0, \qquad u \in L^2_r(a,b),\ \lambda \in \mathbb{C} \setminus \mathbb{R} \end{aligned}$$

We don't need to impose another boundary condition because the fact that $u \in L^2_r(a,b)$ essentially provides a boundary condition.

$$G(x,\xi;\lambda) = \begin{cases} \frac{1}{c} u_1(x) u_2(\xi) & x < \xi \\ \frac{1}{c} u_1(\xi) u_2(x) & x > \xi \end{cases}$$

This need not be a Hilbert-Schmidt kernel. So now we can get a more complicated spectrum. (Recall: the structure of a bounded, self-adjoint operator is entirely real.)

19 2-24-12

19.1 Integral Equations

(Section 4.3 of Logan)

Integral equations arise directly as models ("nonlocal effects"). We can often reformulate differential equations as integral equations.

19.1.1 A Renewal Equation

Problem: Find the birth rate in a population with a known reproduction rate per individual $f(a)$ ($a =$ age) and known survivial rate $s(a)$.

- $u(a,t) =$ population density with respect to age, a, at time t. That is, the total population with age $a \in [a_1, a_2]$ at time t is $\int_{a_1}^{a_2} u(a,t)\,dt$. Equivalently, $u(a,t)\,da =$ the population at time t with age $\in [a, a+da]$.
- $f(a) =$ fecundity
- $s(a) =$ survival rate

We want to find the total birth rate $B(t)$ at time t. Assume that at $t = 0$ we know $u(a,0) = u_0(a)$.

$$\begin{aligned} B(t) &= \int_0^\infty f(a)u(a,t)\,da \\ &= \int_0^t f(a)u(a,t)\,da + \underbrace{\int_t^\infty f(a) \overbrace{u(a,t)}^{=u_0(a-t)s(t)} da}_{\phi(t)} \end{aligned}$$

$$\begin{aligned} u(a-t)\,da &= S(a)B(t-a)\,da \\ B(t) &= \int_0^t f(a)s(a)B(t-a)\,da + \phi(t) \end{aligned}$$

This is a linear *Volterra integral equation*.

19.1.2 Coagulation

(Smolochowski 1916)

Suppose we have a collection of particles of size $0 \le x < \infty$ at time t. They can merge at some known rate $k(x,y)$.

- $n(x,t) =$ (number) density of particles of size x at time t

$$\frac{\partial n}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)n(x-y,t)n(y,t)\,dy - \int_0^\infty K(x,y)n(x,t)n(y,t)$$

This is nonlinear, and it is called an *integro-differential equation*.

Similar example: Boltzmann equation from kinetic theory

- $f(x,v,t) =$ probability density of particles in a gas at position x with velocity v at time t.
- $Q(f) =$ collision term; it is an integral over v

$$f_t + v\frac{\partial f}{\partial x} = Q(f)$$

20 2-27-12

20.1 Reformulation of Differential Equations as Integral Equations

Consider:

1. Initial value problems (IVP's)
2. Eigenvalue problems (EVP's)
3. Boundary value problems (BVP's)
4. Boundary integral equations
    - For example: $\Delta u = 0$ on $\Omega \Leftrightarrow$ integral equation on $\partial\Omega$

20.1.1 IVP's

Consider a first-order scalar IVP:

$$\begin{aligned} \dot{u}(t) &= f(t, u(t)) \\ u(0) &= u_0 \end{aligned}$$

$$u(t) = u_0 + \int_0^t f(s, u(s))\, ds$$

This is a nonlinear Volterra equation. It includes both the ODE and the initial condition.

Picard iteration:

$$u_{n+1}(t) = u_0 + \int_0^t f(s, u_n(s))\, ds, \qquad n = 0, 1, 2, \ldots$$

If $f(t, u)$ is continuous in t and Lipschitz continuous in u, then we can prove that the Picard iterates $\{u_n\}$ converge uniformly to a solution u on a small enough time interval $[0, T]$.

20.1.2 EVP's

$$\begin{aligned} -(pu')' + qu &= \lambda u, \qquad a < x < b \qquad (20.1) \\ u(0) = u(b) &= 0 \end{aligned}$$

Regular SL-EVP. Suppose $\lambda = 0$ is not an eigenvalue. Let $G(x, \xi)$ be the Green's function for $\lambda = 0$. (If $\lambda = 0$ is an eigenvalue, then we could use the Green's function for $\lambda_0 \neq 0$ to "shift" the equation.)

$$\begin{aligned} -(pu')' + qu &= f(x) \\ u(0) = u(b) &= 0 \\ u(x) &= \int_a^b G(x, \xi) f(\xi)\, d\xi \qquad (20.2) \end{aligned}$$

If $u(x)$ is a solution of the EVP (20.1), then

$$u(x) = \lambda \int_a^b G(x, \xi) u(\xi)\, d\xi$$

(Obtained by plugging $f = \lambda u$ into (20.2).) This is a *Fredholm integral equation*.

$$
\begin{aligned}
Ku(x) &= \int_a^b G(x,\xi)u(\xi)\,d\xi \\
Ku &= \mu u, \qquad \mu = \frac{1}{\lambda} \\
Lu &= \lambda u
\end{aligned}
$$

In terms of matrices:

$$
\begin{aligned}
Ax &= \lambda x \\
x &= \lambda A^{-1} x \\
Bx &= \mu x, \qquad \mu = \frac{1}{\lambda},\ B = A^{-1}
\end{aligned}
$$

It turns out that K is a compact, self-adjoint operator on $L^2(a,b)$. So Hilbert spact theory says that it has a complete orthonormal set of eigenfunctions with eigenvalues $|\mu_1| \geq |\mu_2| \geq \ldots \to 0$.

20.1.3 BVP's

$$
\begin{aligned}
-u'' + q(x)u &= f(x), \qquad 0 < x < 1 \\
u(0) = u(1) &= 0
\end{aligned}
$$

We know that we can solve this if $q(x) \geq 0$. If $q(x) < 0$ then we have to worry if 0 is an eigenvalue. $\Rightarrow$ In general we can't solve this explicitly, but we can use approximations.

Suppose $q(x)$ is small, and treat $q(x)u$ as a perturbation:

$$
\begin{aligned}
-u'' &= -qu + f \\
u(0) &= u(1) = 0
\end{aligned}
$$

Let $G(x,\xi)$ be the Green's function for the unperturbed problem:

$$
\begin{aligned}
-u'' &= f(x) \\
u(0) &= u(1) = 0 \\
G(x,\xi) &= \begin{cases} x(1-\xi) & 0 \leq x < \xi \\ \xi(1-x) & \xi \leq x < 1 \end{cases} \\
&= x_<(1 - x_>)
\end{aligned}
$$

Plugging $-qu + f$ into the Green's function representation for u, we get

$$
\begin{aligned}
u(x) &= \int_0^1 G(x,\xi)[-q(\xi)u(\xi) + f(\xi)]\,d\xi \\
u(x) &= -\int_0^1 G(x,\xi)q(\xi)u(\xi)\,d\xi + \underbrace{\int_0^1 G(x,\xi)f(\xi)\,d\xi}_{=g(x)} \\
&= -\int_0^1 K(x,\xi)u(\xi)\,d\xi + g(x), \qquad K(x,\xi) = G(x,\xi)q(\xi)
\end{aligned}
$$

$$
u(x) + \int_0^1 K(x,\xi)u(\xi)\,d\xi = g(x)
$$

This is a Fredholm integral equation of the 2nd kind.

20.2 Neumann Series (or Born Approximation)

For small q, generate approximate solutions by iteration:

$$u + Ku = g,$$
$$\text{where} \quad Ku(x) = \int_0^1 K(x,\xi)u(\xi)\,d\xi = \int_0^1 G(x,\xi)q(\xi)u(\xi)\,d\xi$$

Take $u_0 = g$. Define u_{n+1} by

$$\begin{aligned} u_{n+1} + Ku_n &= g \\ u_{n+1} &= g - Ku_n \\ u_{n+1} &= g - K(g - Ku_{n-1}) \\ &= g - Kg + K^2 u_{n-1} \\ &= g - Kg + K^2 g - K^3 g + \cdots + (-1)^n K^n g \end{aligned}$$

For example,

$$u_2(x) = g(x) - \int_0^1 q(\xi)G(x,\xi)g(\xi)\,d\xi + \int_0^1 q(\xi_2)G(x,\xi_2)\left[\int_0^1 G(\xi_2,\xi_1)q(\xi_1)g(\xi_1)\,d\xi_1\right]d\xi_2$$

21 3-2-12

21.1 Classification of Integral Equations

Suppose $u(x)$ is a complex or real valued function on $a \leq x \leq b$ (for now, think of this interval as finite).

$$Volterra vs. Fredholm \qquad 1st vs. 2nd kind$$

$$\text{Fredholm} \quad \begin{cases} \int_a^b k(x,y)u(y)\,dy = f(x) & \text{1st kind} \\ u(x) - \lambda \int_a^b k(x,y)u(y)\,dy = f(x) & \text{2nd kind} \end{cases}$$

Here, f is a given function on $[a,b]$. $k(x,y)$ (the kernel) is a given function on $x \in [a,b],\ y \in [a,b]$.

$$\text{Volterra} \quad \begin{cases} \int_a^x k(x,y)u(y)\,dy = f(x) & \text{1st kind} \\ u(x) - \lambda \int_a^x k(x,y)u(y)\,dy = f & \text{2nd kind} \end{cases}$$

Note: Volterra equations are a special case of Fredholm equations in which the kernel, $k(x,y)$, is zero for $y > x$.

Hermitial Fredholm equation:

$$k(y,x) = \overline{k(x,y)}$$

(In the real case, this is a symmetric kernel.) It follows that the integral operator $K : L^2(a,b) \to L^2(a,b)$ is given by

$$Ku(x) = \int_a^b k(x,y)u(y)\,dy,$$

and K is self-adjoint in the symmetric case.

$$\begin{aligned}
(Ku, v) &= \int_a^b Ku(x)\overline{v(x)}\,dx \\
&= \int_a^b \int_a^b k(x,y)u(y)\overline{v(x)}\,dx\,dy \\
&= \int_a^b \int_a^b k(y,x)u(x)\overline{v(y)}\,dx\,dy \\
&= \int_a^b u(x) \left(\overline{\int_a^b \overline{k(y,x)}v(y)\,dy} \right) dx \\
&= \int_a^b u(x)K^*v(x)\,dx \\
&= (u, K^*v) \\
K^*v &= \int_a^b \overline{k(y,x)}v(y)\,dy
\end{aligned}$$

The adjoint of K is the integral operator with kernal $\overline{k(y,x)}$. If $k(x,y) = \overline{k(y,x)}$, then $(Ku, v) = (u, Kv)$.

21.2 Degenerate Kernels

$$K(x,y) = \sum_{i=1}^{n} a_i(x)\overline{b_i(y)}$$

Consider the 2nd kind of equation:

$$
\begin{aligned}
u(x) - \lambda \int_a^b k(x,y)u(y)\,dy &= f(x) \\
u - \lambda K u &= f \\
Ku(x) &= \sum_{i=1}^{n} \int_a^b [a_i(x)\overline{b_i(y)}u(y)]\,dy \\
&= \sum_{i=1}^{n} \left[\int_a^b u(y)\overline{b_i(y)}\,dy\right] a_i(x) \\
&= \sum_{i=1}^{n} u_i a_i(x) \\
u_i &= \int_a^b u(y)\overline{b_i(y)}\,dy = (u, b_i) \\
u - \lambda \sum_{i=1}^{n} u_i a_i &= f \\
\Rightarrow \qquad u(x) &= f(x) + \lambda \sum u_i a_i(x) \\
u_i &= (u, b_i) = (f, b_i) + \lambda \sum_{j=1}^{n} u_j \underbrace{(a_j, b_i)}_{=:A_{ij}} \\
u_i - \sum_{j=1}^{n} A_{ij} u_j &= (f, b_i) \\
(I - \lambda A)\mathbf{u} &= \lambda \mathbf{c}, \qquad \mathbf{c} = \begin{pmatrix} (f,b_1) \\ \vdots \\ (f,b_n) \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}
\end{aligned}
$$

Thus, it reduces to an $n \times n$ linear system. We get a unique solution for $\mathbf{u}$ unless $\mu = \frac{1}{\lambda}$ is an eigenvalue of A. In that case, the solution is

$$
u(x) = f(x) + \lambda \sum_{i=1}^{n} u_i a_i(x) + u_h(x),
$$

where $u_h(x)$ is a solution of the homogeneous equation:

$$
u(x) - \lambda \int_a^b k(x,y)u(y)\,dy = 0
$$

Example 21.1.

$$u(x) - \lambda \int_0^1 e^{(x-y)} u(y)\, dy = f(x)$$
$$u(x) - \lambda e^x \int_0^1 e^{-y} u(y)\, dy = f(x)$$
$$u(x) = f(x) + u_1 e^x$$
$$\cancel{f(x)} + u_1 e^x - \lambda e^x \int_0^1 e^{-y} f(y)\, dy - \lambda e^x u_1 \int_0^1 e^{-y} e^y\, dy = \cancel{f(x)}$$

This is a solution provided that

$$u_1 - \lambda \int_0^1 f(y) e^{-y}\, dy - \lambda u_1 = 0$$
$$(1-\lambda) u_1 = \lambda \int_0^1 f(y) e^{-y}\, dy$$

If $\lambda = 1$ is an eigenvalue then the problem is only solvable if $(f, e^{-y}) = 0$. If $\lambda \neq 1$, then

$$u_1 = \frac{\lambda}{1-\lambda} \int_0^1 f(y) e^{-y}\, dy$$

and we get the unique solution

$$u(x) = f(x) + \frac{\lambda}{1-\lambda} e^x \left(\int_0^1 f(y) e^{-y}\, dy \right)$$

If $\lambda = 1$ then we have a solution if $\int_0^1 f(y) e^{-y}\, dy = 0$, in which case

$$u(x) = f(x) + c e^x$$
$$c = \text{ arbitrary constant}$$
$$e^x = \text{ eigenfunction of } K \text{ with eigenvalue 1, since} \quad K(e^x) = \int_0^1 e^{x-y} e^y\, dy = e^x$$

22 3-5-12

22.1 Degenerate Fredholm Equations

$$Ku(x) = \int_a^b k(x,y)u(y)\,dy$$
$$k(x,y) = \sum_{i=1}^n a_i(x)\overline{b_i(y)}, \qquad a_i, b_i \in L^2(a,b)$$

22.1.1 2nd Kind

$$u(x) - \lambda \int_a^b k(x,y)u(y)\,dy = f(x)$$

$f \in L^2(a,b),\ \lambda \in \mathbb{C}$. The solution is

$$u(x) = f(x) + \sum_{i=1}^n u_i a_i(x)$$

where

$$(\mathbf{I} - \lambda \mathbf{A})\mathbf{u} = \lambda \mathbf{c},$$
$$A = \left(A_{ij}\right), \qquad A_{ij} = (a_j, b_i) = \int_a^b a_j(x)\overline{b_i(x)}\,dx$$
$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{C}^n$$
$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{C}^n, \qquad c_i = (f, b_i) = \int_a^b f(x)\overline{b_i(x)}\,dx$$

2 Cases: (Fredholm alternative)

1. $\mu = \frac{1}{\lambda}$ is not an eigenvalue of A. We have a unique solution $u \in L^2(a,b)$ of the 2nd kind equation for every $f \in L^2(a,b)$. There is no nonzero solution of the homogeneous equation with (i.e., $f = 0$).
2. $\mu = \frac{1}{\lambda}$ is an eigenvalue of A. (Then it's also an eigenvalue of K.) We only have a solution for f such that $(\mathbf{I} - \lambda \mathbf{A})\mathbf{u} = \lambda \mathbf{c}$ is solvable. The homogeneous equation has nonzero solutions, and therefore the solution of the nonhomogeneous equation is not unique.

A similar result applies to general 2nd kind Fredholm equations (provided the kernel $k(x,y)$ is not too singular). The moral is that these behave like $n \times n$ linear systems.

$$I - \lambda K = \text{ compact perturbation of the identity}$$
$$(I - \lambda K)u = f$$

22.1.2 1st Kind

$$Ku = f, \qquad k(x,y) = \sum_{i=1}^{n} a_i(x)\overline{b_i(y)}$$

$$\int_a^b k(x,y)u(y)\,dy = f(x)$$

$$\sum_{i=1}^{n} u_i a_i(x) = f(x)$$

$$u_i = \int_a^b u(y)\overline{b_i(y)}\,dy = (u, b_i)$$

We can only solve this if f is a combination of the a_i's,

$$f(x) = \sum_{i=1}^{n} c_i a_i(x),$$

and it has a (particular) solution if and only if there is $u_p \in L^2(a,b)$ such that

$$\int_a^b u_p(x)\overline{b_i(x)}\,dx = c_i, \qquad 1 \le i \le n$$

(assuming that the a_i's are linearly independent). Then the general solution is

$$u(x) = u_p(x)v(x), \qquad \text{where } (v, b_i) = 0, \quad 1 \le i \le n.$$

The moral is that the 1st kind is much nastier than the 2nd kind!

22.2 Spectral Theory

$\mu \in \mathbb{C}$ is an eigenvalue of integral operator $K : L^2(a,b) \to L^2(a,b)$ if $K\phi = \mu\phi$ for some $\phi \in L^2(a,b)$, $\phi \neq 0$. Consider self-adjoint operators with Hermitian kernels: $k(y,x) = \overline{k(x,y)}$. This guarantees that $(Ku,v) = (u,Kv)$.

All eigenvalues of self-adjoint K are real and eigenfunctions with different eigenvalues are orthogonal.

$$\begin{aligned} K\phi &= \mu\phi, \qquad \mu \in \mathbb{C},\ \phi \in L^2(a,b) \\ (K\phi,\phi) &= (\phi, K\phi) \\ (\mu\phi,\phi) &= (\phi,\mu\phi) \\ \mu\|\phi\|^2 &= \overline{\mu}\|\phi\|^2 \\ \mu &= \overline{\mu} \qquad \text{if } \phi \neq 0 \quad \Rightarrow \quad \mu \in \mathbb{R} \end{aligned}$$

If $K\phi_1 = \mu_1\phi_1$ and $K\phi_2 = \mu_2\phi_2$, $\mu_1 \neq \mu_2$, then

$$\begin{aligned} (K\phi_1,\phi_2) &= (\phi_1, K\phi_2) \\ (\mu_1\phi_1,\phi_2) &= (\phi_1,\mu_2\phi_2) \\ \mu_1(\phi_1,\phi_2) &= \mu_2(\phi_1,\phi_2) \\ (\phi_1,\phi_2) &= 0 \end{aligned}$$

Suppose K has a complete orthonormal set of eigenfunctions, $\{\phi_1, \phi_2, \ldots\}$, with eigenvalues $\{\mu_1, \mu_2, \ldots\}$.

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} c_i \phi_i(x) \\
c_i &= (\mu, \phi_i) \\
Ku(x) &= \sum_{i=1}^{\infty} c_i \mu_i \phi_i(x) \\
&= \sum_{i=1}^{\infty} (u, \phi_i)\, \mu_i \phi_i(x) \\
&= \sum_{i=1}^{\infty} \left[\int_a^b u(y) \overline{\phi_i(y)}\, dy \right] \mu_i \phi_i(x) \\
&= \int_a^b u(y) \left[\sum_{i=1}^{\infty} \mu_i \phi_i(x) \overline{\phi_i(y)} \right] dy \\
&= \int_a^b k(x, y) u(y)\, dy \\
k(x, y) &= \sum_{i=1}^{\infty} \mu_i \phi_i(x) \overline{\phi_i(y)}
\end{aligned}$$

This is the eigenfunction expansion of the kernel k, assuming we have a complete orthonormal set of eigenfunctions. Note that the b_i's are the conjugates of the a_i's; this is due to self-adjointness.

$$\int_a^b \int_a^b |k(x, y)|^2\, dx\, dy = \sum_{i=1}^{\infty} \mu_i^2$$

This sum is finite for Hilbert-Schmidt operators.

23 3-7-12

23.1 Hilbert-Schmidt Operators

Definition 23.1. ***Hilbert-Schmidt Operator***

$K : L^2(a,b) \to L^2(a,b)$,

$$Ku(x) = \int_a^b k(x,y)u(y)\,dy$$

We say that K is *Hilbert-Schmidt* if

$$\int_a^b \int_a^b |k(x,y)|^2\,dx\,dy < \infty$$

If $[a,b]$ is a bounded interval and $k(x,y)$ is continuous, then K is Hilbert-Schmidt. K may fail to be Hilbert-Schmidt if

1. it has strong enough singularities

2. $[a,b]$ is unbounded

Example 23.2.

1. $Ku(x) = \frac{1}{x}\int_0^x u(y)\,dy,\ 0 \le x \le 1$

$$k(x,y) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & x < y < 1 \end{cases}$$
$$\int_0^1 |k(x,y)|^2\,dy = \int_0^x \frac{1}{x^2}\,dy = \frac{1}{x}$$
$$\int_0^1 dx \int_0^1 |k(x,y)|^2\,dy = \int_0^1 \frac{1}{x}\,dx = \infty$$

This function is not Hilbert-Schmidt.

2. $Ku(x) = \int_{-\infty}^{\infty} e^{-|x-y|}u(y)\,dy$ on $L^2(-\infty,\infty)$.

$$k(x,y) = e^{-|x-y|}$$
$$\int_{-\infty}^{\infty} |k(x,y)|^2\,dy = \int_{-\infty}^{\infty} e^{-2|x-y|}\,dy$$
$$= \int_{-\infty}^{\infty} e^{-2|t|}\,dt$$
$$\stackrel{?}{=} 1$$
$$\int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} |k(x,y)|^2\,dy\right) = \infty$$

So K is not Hilbert-Schmidt.

3. $k(x,y) = e^{-x^2-y^2}$ on $L^2(-\infty,\infty)$

$$\int_{-\infty}^{\infty} |k(x,y)|^2\,dy = \int_{-\infty}^{\infty} e^{-2x^2}e^{-2y^2}\,dy$$
$$= e^{-2x^2}\tilde{\pi}$$
$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |k(x,y)|^2\,dy = (\tilde{\pi})^2 < \infty$$
$$\tilde{\pi} = \int_{-\infty}^{\infty} e^{-2x^2}\,dx$$

So this is a Hilbert-Schmidt operator.

A Hilbert-Schmidt operator on $L^2(a,b)$ is compact (sufficient compact; not all compact operators are Hilbert-Schmidt). Consider self-adjoint Hilbert-Schmidt operators: $\overline{k(y,x)} = k(x,y),\ \int_a^b\int_a^b |k(x,y)|^2\,dx\,dy < \infty$.

Theorem 23.3.

If K is a self-adjoint, Hilbert-Schmidt operator on $L^2(a,b)$, then

1. K has real eigenvalues $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ such that $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n| \geq \cdots$ (finite multiplicity, except possibly $\mu = 0$), $\mu_n \to 0$ as $n \to \infty$ as $n \to \infty$
2. There is a complete orthonormal set of corresponding eigenfunctions $\phi_1, \phi_2, \ldots, \phi_n, \ldots$, $(\phi_n, \phi_m) = \delta_{nm}$. If $f \in L^2(a,b)$, then

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \qquad c_n = (f_n, \phi_n)$$

$$\left\| f - \sum_{n=1}^{N} c_n \phi_n \right\| \to 0 \qquad \text{as } N \to \infty$$

3.

$$k(x,y) = \sum_{n=1}^{\infty} \mu_n \phi_n(x) \overline{\phi_n(y)},$$

where the series converges in the sense

$$\int_a^b \int_a^b \left| k(x,y) - \sum_{n=1}^{\infty} \mu_n \phi_n(x) \overline{\phi_n(y)} \right|^2 dx\, dy \to 0 \qquad \text{as } N \to \infty$$

Note: (1) and (2) are true for any compact, self-adjoint operator.

23.2 Connection with Sturm-Liouville Problems

$$\begin{cases} Lu = \lambda u, & a < x < b \\ B(u) = 0 \end{cases}$$
$$L = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x)$$
$$B = \text{ self-adjoint BC's}$$

Suppose $\gamma \in \mathbb{R}$ is not an eigenvalue of L (or in its spectrum). Let $G(x,\xi;\gamma)$ be the Green's function for $L - \gamma I$, with BC's B.

$$\begin{cases} (L - \gamma I)u = (\lambda - \gamma)u \\ B(u) = 0 \end{cases}$$
$$u = (\lambda - \gamma)(L - \gamma I)^{-1} u$$
$$(L - \gamma I)^{-1} = k(\gamma)$$
$$k(\gamma)u(x) = \int_a^b G(x,\xi;\gamma) u(\xi)\, d\xi$$
$$u = (\lambda - \gamma) k(\gamma) u$$
$$k(\gamma)u = \left(\frac{1}{\lambda - \gamma} \right) u$$
$$k(\gamma)u = \mu u, \qquad \text{where } \mu = \frac{1}{\lambda - \gamma}$$

If the original SL-EVP has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $k(\gamma)$ has eigenvalues $\mu_n = \frac{1}{\lambda_n - \gamma}$ and the same eigenfunctions, $\phi_n(x)$.

$$\begin{aligned} k(x, \xi; \gamma) &= \sum_{n=1}^{\infty} \mu_n \phi_n(x) \overline{\phi_n(\xi)} \\ &= \sum_{n=1}^{\infty} \frac{\phi_n(x)\overline{\phi_n(\xi)}}{\lambda_n - \gamma} \end{aligned}$$

$k(x, \xi)$ is self-adjoint, since $\gamma \in \mathbb{R}$ and L is self-adjoint. This is the bilinear eigenfunction expansion of the Green's function.

Conclusion: if the Green's function of a SL-EVP is Hilbert-Schmidt, we get a complete set of orthonormal eigenfunctions. This is true in the regular or singular/limit circle case.

Example 23.4.

$$\begin{cases} -u'' = \lambda u, & 0 < x < 1 \\ u(0) = u(1) = 1 \end{cases}$$

$$G(x, \xi_0) = \begin{cases} x(1-\xi) & 0 < x < \xi \\ \xi(1-x) & \xi < x < 1 \end{cases}$$

This is Hilbert-Schmidt with pure point spectrum.

$$\begin{cases} -u'' + u = \lambda u, & -\infty < x < \infty \\ u(x) \to 0 \text{ as } x \to \pm\infty \end{cases}$$

$$G(x, \xi; 0) = \frac{1}{2} e^{-|x-\xi|}$$

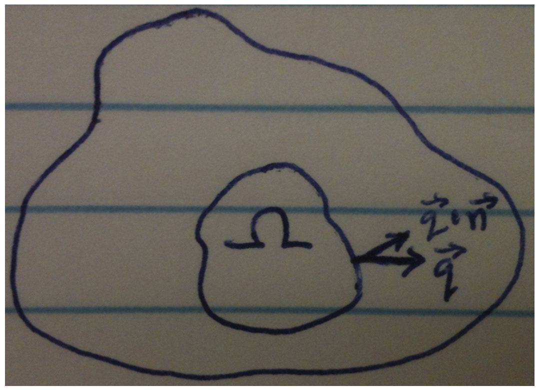
This is not Hilbert-Schmidt, and it has continuous spectrum.

24 3-9-12

24.1 PDEs and Laplace's Equation

(Chapter 6)

Heat Equation:



$$
\begin{aligned}
u(x,t) &= \text{ temperature} \\
e(x,t) &= \text{ thermal energy density/unit volume} \\
\vec{q}(x,t) &= \text{ heat flux vectors} \\
f(x) &= \text{ heat source density} \\
\frac{d}{dx}\underbrace{\int_\Omega e(x,t)\,dx}_{\text{total heat in }\Omega} &= -\int_{\partial\Omega} \vec{q}\cdot\vec{n}\,dS + \int_\Omega f(x,t)\,dx
\end{aligned}
$$

Recall the divergence theorem:

$$
\begin{aligned}
\int_\Omega (\nabla\cdot\vec{q})\,dx &= \int_{\partial\Omega} \vec{q}\cdot\vec{n}\,dS \\
\nabla\cdot\vec{q} &= \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \cdots + \frac{\partial q_n}{\partial x_n} \\
&= \frac{\partial q_i}{\partial x_i} \qquad \text{(summation convention)}
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{d}{dx}\int_\Omega e(x,t)\,dx &= -\int_\Omega (\nabla\cdot\vec{q})\,dx + \int_\Omega f(x,t)\,dx \\
\int_\Omega (e_t + \nabla\cdot\vec{q} - f)\,dx &= 0 \\
e_t + \nabla\cdot\vec{q} &= f \qquad \text{if, say, } e,\ \nabla\cdot\vec{q}, \text{ and } f \text{ are continuous}
\end{aligned}
$$

So we have derived a conservaiton law (or balance law if $f \neq 0$).

$$
\left.\begin{array}{ll}
\text{Fourier's Law:} & \vec{q} = -k\nabla u \\
\text{Energy:} & e = cu
\end{array}\right\}\text{constitutive relations}
$$

k is the thermal conductivity (isotropic material). This is saying that heat flows in the opposite direction to the temperature gradient.

$$
\begin{aligned}
\nabla \cdot (\nabla u) &= \Delta u = \nabla^2 u \\
cu_t - k\Delta u &= f \\
u_t &= \nu \Delta u + f(x),
\end{aligned}
$$

where $f \leftarrow \frac{1}{c} f$ and $\nu = \frac{k}{c}$ is the thermal diffusivity with units $\frac{\mathrm{L}^2}{\mathrm{T}}$.

24.1.1 Steady Temperature

If $u = u(x)$ independent of t ($\nu = 1$ by non-dimensionalization), then

$$
\begin{aligned}
-\Delta u = f(x) \qquad & \text{in } \Omega \\
u = 0 \qquad & \text{on } \partial\Omega
\end{aligned}
$$

This is a Dirichlet problem for the body Ω with heat sources f and the boundary held at 0 temperature.

Now consider

$$
\begin{aligned}
-\Delta u = f(x) \qquad & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 \qquad & \text{on } \partial\Omega
\end{aligned}
$$

This is a Neumann problem for Δ (insulated boundary).

24.1.2 Separation of Variables

$$
\begin{aligned}
u_t = \Delta u, \qquad & x \in \Omega,\ t > 0 \\
u = 0, \qquad & x \in \partial\Omega, t > 0 \\
u(x, 0) = u_0(x) &
\end{aligned}
$$

Let's look for separated solutions:

$$u(x, t) = v(x)T(t)$$

Then

$$
\begin{aligned}
u_t &= v\dot{T} \\
\Delta u &= T\Delta v \\
v\dot{T} &= T\Delta v \\
\frac{\Delta v}{v} &= \frac{\dot{T}}{T} = -\lambda \\
T(t) &= e^{-\lambda t} \qquad \text{constant will be absorbed into } v \\
u(x, t) &= v(x)e^{-\lambda t} \\
&\begin{cases} -\Delta v = \lambda v \qquad \text{in } \Omega \\ v = 0 \qquad \text{on } \partial\Omega \end{cases}
\end{aligned}
$$

So λ is an eigenvalue of $-\Delta$ with Dirichlet BC's. Suppose we have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with a complete set of eigenfunctions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$. That is, $-\Delta\phi_n = \lambda_n \phi_n,\ \phi_n = 0$ on $\partial\Omega$. The general solution of the PDE + BC's is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \phi_n(x)$$

Now all that's left is to satisfy the IC. We choose the constants c_n such that

$$\begin{aligned} u_0(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\ c_n &= \int_{\Omega} u_0(x) \phi_n(x)\, dx \end{aligned}$$

25 3-12-12

25.1 Green's Identities

Let Ω be a bounded domain with smooth boundary $\partial\Omega$. If $u, v \in C^1(\overline{\Omega})$, then

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \eta} v\right) ds = \int_\Omega (\Delta u v + \nabla u \cdot \nabla v)\, dA \qquad \left(\frac{\partial u}{\partial \eta} v = \nabla u \cdot \eta\right) \tag{25.1}$$

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \eta} v - \frac{\partial v}{\partial \eta} u\right) ds = \int_\Omega (v\Delta u - u\Delta v)\, dA \tag{25.2}$$

Note: (25.2) is the multidimensional version of $uv'' - vu'' = (uv' - vu')'$.

Proof. (25.2) is a consequence of (25.1).

$$\int_{\partial\Omega} (\vec{F} \cdot \vec{\eta})\, ds = \int_\Omega (\operatorname{div} \vec{F})\, dA$$

Recall:

$$\vec{F} = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix}, \qquad \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

So

$$\operatorname{div} \vec{F} = \nabla u \cdot \nabla v + u\Delta v$$

$$\int_{\partial\Omega} u(\nabla v \cdot \eta)\, ds = \int_\Omega (\nabla u \cdot \nabla v + u\Delta v)\, da$$

□

We will be studying the problem

$$\begin{aligned} \Delta u &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{25.3}$$

We will split this into 2 pieces:

$$\begin{aligned} \Delta v &= 0 \qquad \text{in } \Omega \\ v &= g \qquad \text{on } \partial\Omega \end{aligned} \qquad\qquad \begin{aligned} \Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Each of these is homogeneous in a sense. Today, we will focus on the 2nd problem.

Theorem 25.1.

If $u, v \in C^1(\overline{\Omega})$ and u, v satisfy (25.3), then $u = v$ on $\overline{\Omega}$.

Proof. Let $w = u - v$. Then $\Delta w = 0$ in Ω, and $w = 0$ on $\partial\Omega$. Let's use the first Green's identity, (25.1).

$$\int_{\partial\Omega} \frac{\partial w}{\partial \eta} \underbrace{w}_{=0}\, ds = \int_\Omega (w \underbrace{\Delta w}_{=0} + \nabla w \cdot \nabla w)\, dA = 0$$

$$\int_\Omega \nabla w \cdot \nabla w\, dA = \int_\Omega |\nabla w|^2\, dA$$

$$\nabla w = 0 \qquad \text{in } \Omega \quad \Rightarrow \quad w = 0 \qquad \text{in } \Omega$$

□

So how do we solve this problem?

$$\begin{aligned} \Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

We will solve it via eigenfunction expansion:

$$\begin{aligned} \Delta u &= \lambda u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

So we want to find

$$\begin{aligned} \Delta u_j &= \lambda_j u_j \\ \langle u_i, u_j \rangle &= \delta_{ij} \\ \{u_j\} &\text{ is complete.} \end{aligned}$$

If we have an eigenfunction basis, then we can rewrite

$$u = \sum \alpha_j u_j, \qquad f = \sum \beta_j u_j.$$

Then

$$\begin{aligned} \Delta u = \sum \alpha_j \Delta u_j &= \sum \beta_j u_j \\ \sum \alpha_j \lambda_j u_j &= \sum \beta_j u_j \\ \alpha_j \lambda_j &= \beta_j \\ \alpha_j &= \frac{\beta_j}{\lambda_j} \end{aligned}$$

So now we direct our attention to this problem:

$$\begin{aligned} \Delta u &= \lambda u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

We want and expect

$$\langle \Delta u, v \rangle = \langle u, \Delta v \rangle .$$

25.2 Some Properties

1. Self-adjoint. If $u, v \in C^1(\overline{\Omega})$ and $u, v = 0$ on $\partial\Omega$, then

$$\langle \Delta u, v \rangle = \langle u, \Delta v \rangle .$$

Proof.

$$\begin{aligned} \langle \Delta u, v \rangle - \langle u, \Delta v \rangle &= \int_\Omega (\Delta u \overline{v} - u \Delta \overline{v})\, dA \\ &= \int_{\partial\Omega} \left(\frac{\partial u}{\partial \eta} \overline{v} - u \frac{\partial \overline{v}}{\partial \eta} \right) ds \\ &= 0 \end{aligned}$$

□

2. Real eigenvalues.

$$\begin{aligned} \lambda \langle u, u \rangle &= \langle \lambda u, u \rangle = \langle \Delta u, u \rangle \\ &= \langle u, \Delta u \rangle = \langle u, \lambda u \rangle = \overline{\lambda} \langle u, u \rangle \end{aligned}$$

3. Orthogonality of eigenspaces. If $\Delta u = \lambda u,\ \Delta v = \eta v,\ \eta \neq \lambda$, then $\langle u, v \rangle = 0$.

4. Δ is negative definite. $\langle \Delta u, u \rangle < 0$. Thus, all the eigenvalues are negative.

   Proof. Use Green's identity #1, (25.1).

$$\begin{aligned} 0 = \int_{\partial\Omega} \left(\frac{\partial \overline{u}}{\partial \eta} u \right) ds &= \int_{\Omega} (\overline{u} \Delta u + \nabla u \cdot \nabla \overline{u}) \, dA \\ 0 &= \langle u, \Delta u \rangle + \underbrace{\int_{\Omega} |\nabla u|^2 \, dA}_{>0} \\ 0 &> \langle u, \Delta u \rangle \end{aligned}$$

□

Consider the problem

$$\begin{aligned} \Delta u &= f \qquad \text{in } \Omega = [0,1] \times [0,1] \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

How do we find eigenvalues:

$$\begin{aligned} \Delta u &= \lambda u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

From the book, plug in a formula $u(x, y) = f(x)g(y)$. Use this to compute an eigenfunction basis.

26 3-14-12

26.1 Vibrations of a Drum

The vertical displacement of a membrane is given by

$$z = u(x, y, t).$$

It satisfies the wave equation:

$$\begin{aligned} u_{tt} &= c_0^2 \Delta u \\ \Delta u &= u_{xx} + u_{yy} \\ c_0 &= \text{ constant (wave speed)} \end{aligned}$$

IBVP:

$$\begin{aligned} u_{tt} = c_0^2 \Delta u \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \\ u(x,0) = f(x) \qquad & t = 0 \\ u_t(x,0) = g(x) \end{aligned}$$

Look at separated solutions:

$$u(x, y, t) = v(x, y)e^{-i\omega t}.$$

Plugging this in to the wave equation, we have

$$\begin{aligned} -\omega^2 v &= c_0^2 \Delta v \\ -\Delta v &= k^2 v, \qquad k^2 = \frac{\omega^2}{c_0^2} \\ v &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

We get nontrivial solutions if $k^2 = \lambda_n \;\Leftrightarrow\; \omega^2 = c_0^2 \lambda_n$, where

$$\begin{aligned} -\Delta v = \lambda_n v \qquad & \text{in } \Omega \\ v = 0 \qquad & \text{on } \partial\Omega. \end{aligned}$$

26.2 Examples of Eigenvalues of the Laplacian

Consider a rectangular domain: $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$.

$$\begin{aligned} -\Delta u = \lambda u, \qquad & 0 < x < a,\ 0 < y < b \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

Separation of variables:

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ -(u_{xx} + u_{yy}) &= \lambda u \\ -(X''Y + XY'') &= \lambda XY \\ -\frac{X''}{X} - \frac{Y''}{Y} &= \underbrace{\lambda}_{>0} \\ -\frac{X''}{X} = p, \qquad -\frac{Y''}{Y} &= q, \qquad p + q = \lambda \\ X'' + pX = 0, \qquad & X(0) = X(a) = 0 \\ Y'' + qY = 0, \qquad & Y(0) = Y(b) = 0 \end{aligned}$$

The crucial thing that lets us solve this problem is that we can find separable solutions of the Laplacian that are appropriate for the boundary conditions.

$$\begin{aligned}
X &= \sin\left(\frac{m\pi x}{a}\right), \qquad p = \frac{m^2\pi^2}{a^2} \\
Y &= \sin\left(\frac{n\pi y}{b}\right), \qquad q = \frac{n^2\pi^2}{b^2} \\
\lambda &= p + q \\
u_{m,n}(x,y) &= \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \\
\lambda_{m,n} &= \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}, \qquad m, n = 1, 2, 3, \ldots
\end{aligned}$$

Because these X's and Y's form a complete set, we can argue that there are no other eigenfunctions. (Note: the multiplicity of an eigenvalue is a number theory question.)

Example 26.1. ***Laplacian on a Circle***

Let Ω be a circle of radius a.

$$\begin{aligned} -\Delta u &= \lambda u, \qquad r < a \\ u &= 0, \qquad r = a \end{aligned}$$

The Laplacian in polar coordinates is

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Separation of variables:

$$\begin{aligned}
u(r,\theta) &= R(r)T(\theta) \\
T(\theta) &= e^{in\theta}, \qquad n \in \mathbb{Z} \\
-\Delta u &= \lambda u \\
-\left[\frac{1}{r}(rR')'T + \frac{1}{r^2}RT''\right] &= \lambda RT \\
-\left[\frac{(rR')'}{rR} + \frac{1}{r^2}\frac{T''}{T}\right] &= \lambda \\
-\frac{r(rR')'}{R} - \frac{T''}{T} &= \lambda r^2 \\
&\begin{cases} -T'' = cT \\ T(0) = T(2\pi) \\ T'(0) = T'(2\pi) \end{cases} \\
T'' + \underbrace{n^2}_{=c} T &= 0 \\
-\frac{(rR')'}{rR} + \frac{n^2}{r^2} &= \lambda \\
&\begin{cases} -(rR')' + \frac{n^2}{r}R = \lambda r R, \qquad 0 < r < a \\ R(a) = 0 \\ rR'(r) \to 0 \qquad \text{as } r \to 0 \quad (\text{or } R(r) \text{ is bounded as } r \to 0) \end{cases} \\
z &= \sqrt{\lambda} r, \qquad (\text{we know } \lambda > 0) \\
\frac{d}{dr} &= \sqrt{\lambda}\frac{d}{dz} \\
-\sqrt{\lambda}\frac{d}{dz}\left(\sqrt{\lambda} r \frac{dR}{dz}\right) + \frac{n^2}{r}R &= \lambda r R \\
-\frac{d}{dz}\left(z\frac{dR}{dz}\right) + \frac{n^2}{z}R &= zR \qquad \text{Note: no } \lambda \text{ dependence}
\end{aligned}$$

This is Bessel's equation of order n. The solution is bounded at $r = 0$ is denoted $J_n(z) =$ Bessel function of order n. $J_n(z)$ has infinitely many positive zeros; let $j_{n,k}$ denote the kth zero of $J_n(z)$.

Example 26.2. ***Laplacian on a Circle (Continued)***

We want

$$
\begin{aligned}
R(a) &= 0 \\
R(r) &= J_n(\sqrt{\lambda} r) \\
J_n(\sqrt{a}) &= 0 \\
\sqrt{\lambda} a &= j_{n,k}, \qquad n = 0, 1, 2, \ldots, \quad k = 1, 2, 3, \ldots
\end{aligned}
$$

For example, with $n = 0$ we have

$$
\begin{aligned}
u &= J_0(\sqrt{\lambda_{0,k}} r) \\
\sqrt{\lambda} a &= j_{0,k}
\end{aligned}
$$

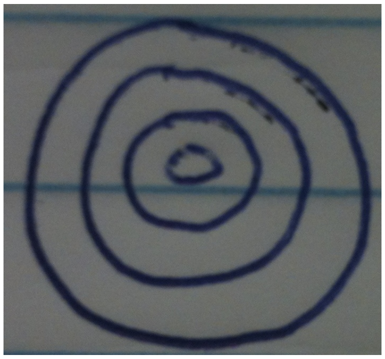


Figure 5: $n = 0$.

With $n = 1$, we have

$$u = J_1(\sqrt{\lambda_{1,k}} r)$$

Figure 6: $n = 1$.

With $n = 2$, we have

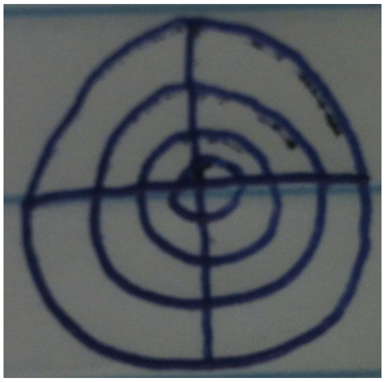


Figure 7: $n = 2$.

27 3-16-12

Extra office hours on Tuesday 2-3:30

Example 27.1. ***M. Kac***

Can you hear the shape of a drum? (1966)

Suppose you know the Laplacian eigenvalues. Can you determine the region?

Gordon, Webb, Wolpert (1992): in 2-D, no!

27.1 Potential Theory

Suppose we have a force field $\vec{E}(x)$ with sources $\rho(x)$.

1. Assume $\vec{E}$ is conservative: $\vec{E} = -\nabla\phi$
2. Source equation: div $\vec{E} = \rho$

Putting these together, we get the Poisson equation:

$$-\Delta\phi = \rho.$$

1\.

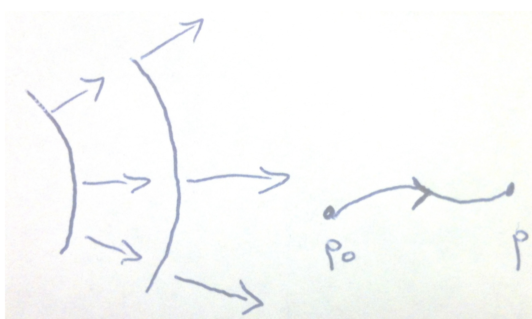


Figure 8: $\phi =$ constant.

The work done against the force field moving from p_0 to p is

$$\begin{aligned} -\int_c \vec{E}\cdot d\vec{x} &= \int_c \nabla\phi\cdot d\vec{x} \\ &= \phi(p) - \phi(p_0) \\ \phi(p) &= \phi(p_0) + \text{ work done against } \vec{E}\ (p_0 \to p) \end{aligned}$$

The work done is independent of the curve.

2.

$$
\begin{aligned}
\operatorname{div} \vec{E} &= \rho \\
\int_{\Omega} (\operatorname{div} \vec{E})\, dx &= \int_{\Omega} \rho\, dx \\
\int_{\partial\Omega} \vec{E} \cdot \vec{n}\, dx &= \int_{\Omega} \rho\, dx
\end{aligned}
$$

$$\text{flux of } \vec{E} \text{ through } \partial\Omega \;=\; \text{total charge inside } \Omega$$

Example 27.2.

1. Electrostatics: $\vec{E} =$ electric field, $\rho =$ charge density
2. Gravity (Newton): $\vec{E} =$ gravitational field, $\rho =$ mass density

27.2 Free Space Green's Function

$$
\begin{aligned}
-\Delta G &= \delta(x) \qquad \text{in } \mathbb{R}^n \\
G(x) &= \text{ potential due to a point source at the origin}
\end{aligned}
$$

Note:

$$
\begin{aligned}
-\Delta G(x,\xi) &= \delta(x-\xi) \\
G(x,\xi) &= G(x-\xi)
\end{aligned}
$$

Recall:

$$
\begin{aligned}
-G'' + G &= \delta(x-\xi), \qquad -\infty < x < \infty \\
G(x,\xi) &= \frac{1}{2} e^{-|x-\xi|}
\end{aligned}
$$

Back to our system:

$$-\Delta u = f(x), \qquad x \in \mathbb{R}^n$$

Idea:

$$
\begin{aligned}
f(x) &= \int \delta(x-\xi) f(\xi)\, d\xi \\
u(x) &= \int G(x-\xi) f(\xi)\, d\xi
\end{aligned}
$$

Thus, we represent our source as a superposition of point sources and solve via the Green's function. Formally:

$$
\begin{aligned}
-\Delta u(x) &= -\Delta \int G(x-\xi) f(\xi)\, d\xi \\
&= \int (-\Delta G) f(\xi)\, d\xi \\
&= \int \delta(x-\xi) f(\xi)\, d\xi \\
&= f(x)
\end{aligned}
$$

This is completely analogous to the Green's function representation we used in the ODE case.

27.3 δ-function in $\mathbb{R}^n$

Formally:

$$\delta(x) = 0, \qquad x \neq 0$$
$$\int \delta(x)\, dx = 1$$

Approximate the δ function by functions that spike at the origin and have unit integral.

Example 27.3.

$$\delta_\epsilon(x) = \begin{cases} c & |x| < \epsilon \\ 0 & |x| > \epsilon \end{cases}$$
$$\int \delta_\epsilon(x) = 1 \qquad \text{(by correctly choosing } c\text{)}$$
$$c \cdot \mathrm{Vol}(B_\epsilon) = 1$$
$$n = 2: \qquad c \cdot \pi\epsilon^2 = 1$$
$$\delta_\epsilon(x) = \begin{cases} \frac{1}{\pi\epsilon^2} & |x| < \epsilon \\ 0 & |x| > \epsilon \end{cases}$$

$$n = 3: \qquad c \cdot \frac{4}{3}\pi\epsilon^3 = 1$$
$$\delta_\epsilon(x) = \begin{cases} \frac{3}{4\pi\epsilon^3} & |x| < \epsilon \\ 0 & |x| > \epsilon \end{cases}$$

27.4 Free-Space Green's Function

$$-\Delta G = \delta(x)$$

1. $\Delta G = 0, \quad x \neq 0$
2.

$$B_\epsilon := \{x \mid |x| \leq \epsilon\}$$
$$\int_{B_\epsilon} \Delta G\, dx = \int_{B_\epsilon} \delta(x)\, dx$$
$$-\int_{\partial B_\epsilon} \frac{\partial G}{\partial n}\, dS = 1 \qquad \text{(Divergence Theorem)}$$
$$\int_{\partial B_\epsilon} \frac{\partial G}{\partial n}\, dS = -1 \qquad \forall\, \epsilon > 0$$

We expect the solution to be spherically symmetric. After all, the Laplacian is rotationally invariant. So we look for solutions $G = G(r)$, where $r = |x|$.

$$\begin{aligned}
\Delta G &= \frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1}\frac{dG}{dr}\right) \\
&= 0 \qquad r > 0 \\
\frac{d}{dr}\left(r^{n-1}\frac{dG}{dr}\right) & \\
r^{n-1}\frac{dG}{dr} &= c \\
\frac{dG}{dr} &= \frac{c}{r^{n-1}} \\
G(r) &= \begin{cases} \frac{c'}{r^{n-2}} & n \geq 3 \\ c' \log r & n = 2 \end{cases} \\
\int_{\partial B_\epsilon} \frac{\partial G}{\partial r}\, dS &= -1 \\
n = 2: \qquad \int_{\partial B_\epsilon} \frac{c}{r}\, dS &= -1 \\
\frac{c}{\epsilon}\int_{\partial B_\epsilon} dS &= -1 \\
\frac{c}{\epsilon} \cdot 2\pi\epsilon &= -1 \\
c &= -\frac{1}{2\pi} \\
G(x) &= -\frac{1}{2\pi} \log|x|
\end{aligned}$$

$$\begin{aligned}
n = 3: \qquad \int_{\partial B_\epsilon} dS &= -1 \\
\int_{\partial B_\epsilon} -\frac{c}{r^2}\, dS &= -1 \\
\frac{c}{\epsilon^2}\underbrace{\int_{\partial B_\epsilon} dS}_{4\pi\epsilon^2} &= 1 \\
c &= \frac{1}{4\pi} \\
G(x) &= \frac{1}{4\pi|x|}
\end{aligned}$$

28 3-19-12

28.1 Green's Function for Laplace's Equation on Bounded Domains

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

Eigenfunction expansion:

$$
\begin{aligned}
-\Delta \phi_n &= \lambda_n \phi_n \qquad \text{in } \Omega, \qquad 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots \\
\phi_n &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

$\{\phi_n(x) \mid n = 1, 2, \ldots\}$ is a complete (real) orthonormal set in $L^2(\Omega)$.

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\
f(x) &= \sum_{n=1}^{\infty} f_n \phi_n(x) \\
c_n &= \int_\Omega u(x)\phi_n(x)\,dx \\
f_n &= \int_\Omega f(x)\phi_n(x)\,dx \\
-\Delta u &= \sum_{n=1}^{\infty} \lambda_n c_n \phi_n \\
&= \sum_{n=1}^{\infty} f_n \phi_n \\
\lambda_n c_n &= f_n \\
c_n &= \frac{f_n}{\lambda_n}
\end{aligned}
$$

$\lambda = 0$ is not an eigenvalue of this equation. This follows from the energy condition.

However, for the Neumann problem:

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

$\lambda = 0$ is an eigenvalue with $\phi_0 = 1$. This equation is solvable if

$$
(1, f) = \int_\Omega f\,dx = 0.
$$

This means that there is no net heat generation.

Back to our Dirichlet system... The solution is

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n} \phi_n(x) \\
u(x) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_\Omega f(\xi)\phi_n(\xi)\, d\xi \right) \phi_n(x) \\
&= \int_\Omega G(x,\xi) f(\xi)\, d\xi \\
G(x,\xi) &= \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}
\end{aligned}
$$

This is the bilinear expansion of the Green's function.

More generally:

$$
\begin{aligned}
-\Delta u &= \lambda u + f(x) \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

$$
\begin{aligned}
u(x) &= \int_\Omega G(x,\xi;\lambda) f(\xi)\, d\xi \\
G(x,\xi;\lambda) &= \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n - \lambda}
\end{aligned}
$$

Thus, the eigenvalues are shifted by λ.

Example 28.1.

$$
\begin{aligned}
-\Delta u &= f(x) \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega \\
\Omega &= (0,1) \times (0,1)
\end{aligned}
$$

$$
\begin{aligned}
\phi_{m,n}(x,y) &= 2\sin(m\pi x)\sin(n\pi y) \\
\lambda_{m,n} &= \pi^2(m^2+n^2) \\
G(\underbrace{x,\xi}_{x \to (x,y)} \; ; \; \underbrace{\xi,\eta}_{\xi \to (\xi,\eta)}) &= \frac{4}{\pi^2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin(m\pi x)\sin(n\pi y)\sin(m\pi\xi)\sin(n\pi\eta)}{m^2+n^2}
\end{aligned}
$$

Note: $G(x,\xi) = G(\xi,x)$. Thus, G is symmetric and self-adjoint.

28.2 Representation in Terms of Free Space Green's Function

$G(x, \xi)$ is the solution of

$$-\Delta G = \delta(x - \xi) \qquad x \in \Omega$$
$$G = 0 \qquad x \in \partial\Omega$$

$$G_F(x - \xi) = \begin{cases} -\frac{1}{2\pi}\log|x - \xi| & n = 2 \text{ dimensions} \\ \frac{1}{4\pi|x-\xi|} & n = 3 \text{ dimensions} \end{cases}$$

G_F is the free space Green's function.

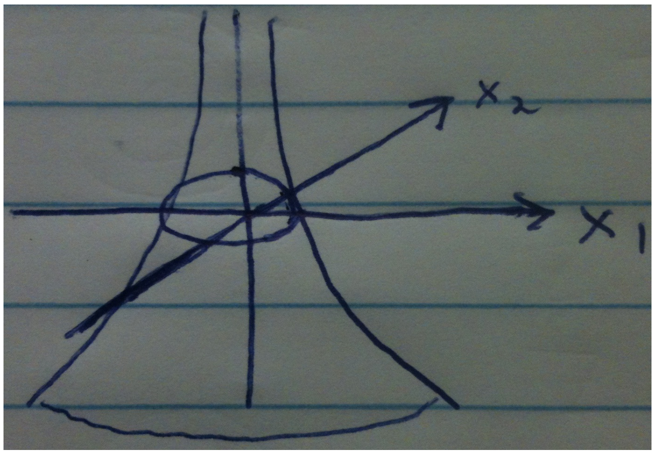


Figure 9: $n = 2$.

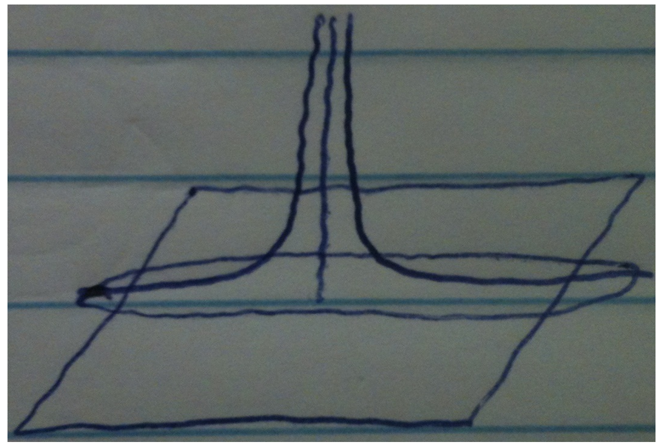


Figure 10: $n = 3$.

$$G(x, \xi) = G_F(x - \xi) + \phi(x; \xi)$$
$$\Delta\phi = 0 \qquad \text{in } \Omega$$
$$\phi(x; \xi) = -G_F(x - \xi) \qquad x \in \partial\Omega$$

where $\phi(x; \xi)$ is a harmonic function (the solution of $\Delta\phi = 0$). ϕ cancels out the value of G_F on the boundary.

28.3 Green's Formula

$$-\Delta G = \delta(x - \xi) \qquad x \in \Omega \qquad\qquad (\Delta \text{ is the Laplacian wrt } x)$$
$$G = 0 \qquad x \in \partial\Omega$$

We want to solve

$$-\Delta u = f(x) \qquad x \in \Omega$$
$$u = 0 \qquad x \in \partial\Omega$$

$$\int_\Omega [u(x)\Delta G(x, \xi) - G(x, \xi)\Delta u(x)]\, dx = \int_{\partial\Omega} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS(x)$$
$$\int_\Omega [-u(x)\delta(x - \xi) + G(x, \xi)f(x)]\, dx = 0$$
$$-u(\xi) + \int_\Omega \underbrace{G(x, \xi)}_{=G(\xi,x)} f(x)\, dx = 0$$
$$u(x) = \int_\Omega G(x, \xi)f(\xi)\, d\xi \qquad\qquad (\text{Rename: } \xi \to x,\ x \to \xi)$$

Since u and G satisfy the BC's, they cancel out, as in the SL problem.

Index