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1.1 Dimensional Analysis

We have a fundamental system of units: $(d_1, d_2, \dots, d_r)$.

Example 1.1. ***Mechanics***

- mass M
- length L
- time T

Derived units, e.g. velocity $V = \frac{L}{T}$ and acceleration $A = \frac{L}{T^2}$. We can use different sets of units as fundamental units (provided they're independent). For example, we could use mass, velocity, and acceleration. Any model of a system must be invariant under rescalings that correspond to changes in the system of units.

Let's say we have a fundamental system of (independent) units: $d_1, d_2, \dots, d_r$. We have a set of quantities in the model:

$$\begin{cases} a_1, a_2, \dots, a_r & \text{with dimension } [a_i] = d_i \\ \vdots & \\ b_{r+1}, \dots, b_n & \end{cases}$$

Let's say b_j has dimensions

$$[b_j] = d_1^{\beta_{1j}} d_2^{\beta_{2j}} \cdots d_r^{\beta_{rj}}.$$

Then the model can only depend on

$$\Pi_j = \frac{b_j}{a_1^{\beta_{1j}} a_2^{\beta_{2j}} \cdots a_r^{\beta_{rj}}}.$$

So our model has:

- r independent dimensions
- n independent quantities

Then dimensional analysis says it depends on $n - r$ dimensionless variables. (This is called the Buckingham Pi Theorem.)

1.2 Fluids Flows, Reynold's Number

Let's say we have a sphere in a flow. What is the drag on the sphere?

Parameters:

- $u =$ speed of the fluid, $[u] = \frac{L}{T}$
- $d =$ diameter of the sphere, $[d] = L$
- $\mu =$ viscosity of the fluid, $[\mu] = \frac{M}{LT}$
- $\rho_0 =$ density of the fluid, $[\rho_0] = \frac{M}{L^3}$
- Assume the fluid is incompressible (this is OK if $u \ll c_0$, the speed of sound in the fluid)

Fundamental units: M, L, T.

In a Newtonian fluid:

- T = viscous stress tensor,

  $$T = \mu(\nabla u + \nabla u^T),$$

  where u = velocity. This gives the force/unit area. The dimensions of T are

  $$[T] = \frac{ML}{T^2} \cdot \frac{1}{L^2} = \frac{M}{LT^2}$$
  $$[\nabla u] = \frac{1}{T}$$
  $$[\mu] = \frac{M}{LT}$$

We define the kinematic viscosity:

$$\nu = \frac{\mu}{\rho_0}$$
$$[\nu] = \frac{L^2}{T}$$

The physical interpretation of this quantity is diffusivity of momentum.

$$\nu \approx 1\text{mm}^2/\text{s in water}$$
$$\nu \approx 15\text{mm}^2/\text{s in air}$$

We can define the Reynold's number:

$$\text{Re} = \frac{ud}{\nu}.$$

This is the crucial dimensionless parameter that controls everything.

Back to our question about drag on a sphere. D = drag force with dimensions $[D] = \frac{ML}{T^2}$.

$$[\rho_0 u^2 d^2] = \frac{M}{L^3} \cdot \frac{L^2}{T^2} \cdot L^2 = \frac{ML}{T^2}$$
$$\frac{D}{\rho_0 u^2 d^2} = F(\text{Re})$$
$$D = \rho_0 u^2 d^2 F(\text{Re})$$

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2.1 Navier-Stokes Equation

$$\rho_0(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}) + \nabla p = \mu_0\Delta\vec{u}$$
$$\nabla\cdot\vec{u} = 0$$

- $\vec{u} = \vec{u}(\vec{x}, t)$ is the fluid velocity
- $p = p(\vec{x}, t)$ is the pressure
- $\vec{u} = (u_1, u_2, u_3)$
- $\vec{x} = (x_1, x_2, x_3)$
- $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

Parameters

- ρ_0 = fluid density
- μ = fluid viscosity
- U = "typical" flow velocity
- L = "typical" flow length scale

Dimensionless variables

- $\vec{u}^* = \frac{\vec{u}}{U}$
- $\vec{x}^* = \frac{\vec{x}}{L}$
- $t^* = \frac{Ut}{L}$
- $p^* = \frac{p}{\rho_0 U^2}$
  - $[\nabla p] = [\rho_0 \vec{u}_t]$
  - $\frac{[p]}{L} = [\rho_0]\frac{L}{T^2}$
  - $[p] = [\rho_0]\frac{L^2}{T^2}$
- $\nabla = \frac{1}{L}\nabla^*$
- $\partial_t = \frac{dt^*}{dt}\partial_{t^*} = \frac{U}{L}\partial_{t^*}$

$$\rho_0\left[\frac{U}{L}(U\vec{u}^*)_{t^*} + \frac{U^2}{L}\vec{u}^*\cdot\nabla^*\vec{u}^*\right] + \frac{\rho_0 U^2}{L}\nabla^* p^* = \frac{\mu U}{L^2}\Delta^*\vec{u}^*$$
$$\vec{u}^*_{t^*} + \vec{u}^*\cdot\nabla^*\vec{u}^* + \nabla^* p^* = \frac{1}{\mathrm{Re}}\Delta^*\vec{u}^*$$
$$\nabla^*\cdot\vec{u}^* = 0$$

2.2 Low Reynolds Number Flows (Re $\to 0$)

$$
\begin{aligned}
p^* &= \frac{\tilde{p}}{\mathrm{Re}} \\
\tilde{p} &= \mathrm{Re} \cdot p^* = \frac{UL}{\nu} \cdot \frac{p}{\rho_0 U^2} = \frac{L}{\mu U} p
\end{aligned}
$$

As Re $\to 0$, we get Stokes equations:

$$
\begin{aligned}
\nabla^* \tilde{p} &= \Delta^* \vec{u}^* \\
\nabla^* \cdot \vec{u}^* &= 0.
\end{aligned}
$$

These are linear!

Example 2.1. ***Drag on a Sphere as Re*** $\to 0$

$$
D = \rho_0 U^2 L^2 F(\mathrm{Re})
$$

Consider $\lim_{\mathrm{Re} \to 0} D$. Since the drag is linear in U, we need

$$
\begin{aligned}
F(\mathrm{Re}) &= \frac{c}{\mathrm{Re}} \\
D &= \rho_0^2 U^2 L^2 \cdot \frac{c}{\mathrm{Re}} = c \frac{\rho_0 U^2 L^2 \nu}{UL} = c \mu_0 UL
\end{aligned}
$$

Stokes (1851):

$$
D = 6\pi \mu_0 a U,
$$

where a is the radius of a sphere.

2.3 High Reynolds Number Limit (Re $\to \infty$)

Formally, we get the Euler equations.

$$
\begin{aligned}
\vec{u}^*_{t^*} + \vec{u}^* \cdot \nabla \vec{u}^* + \nabla^* p^* &= 0 \\
\nabla^* \cdot \vec{u}^* &= 0
\end{aligned}
$$

This is nonlinear!

Turbulence, Prandtt boundary layer term $\to$ singular perturbation neglecting higher derivatives

2.4 Similarity Solutions

Consider the heat flow due to a point source.

$$
\begin{aligned}
u_t &= v \Delta u \\
u(x, 0) &= E\delta(x)
\end{aligned}
$$

$u(x, t) =$ temperature of (infinite) body. Inject total heat energy E at $x = 0$ at $t = 0$.

- θ = temperature dimension, $[u] = \theta$
- L = length, $[x] = L$
- T = time, $[t] = T$

Parameters ν, E

- $[\nu] = \frac{L^2}{T}$
- $[E] = \theta L^n$
  - At $t = 0$, $\int u\,dx = \int E\delta(x)\,dx = E$
  - $[E] = [\int u\,dx] = \theta L^n$

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3.1 Heat Equation

$$u_t = \nu \Delta u$$
$$u(x,0) = E\delta(x)$$

$u(x,t)$ is the temperature, $x \in \mathbb{R}^n$.

Parameters

- ν: thermal diffusivity, $[\nu] = \frac{L^2}{T}$
- E: initial heat, $[E] = \theta L^n$

Dependent variables: u ($[u] = \theta$).

Independent variables: r ($[r] = L$), t ($[t] = T$).

So we have

- 5 quantities: ν, E, u, r, t
- 3 dimensions: θ, L, T

We can form 2 dimensionless quantities.

- Time: t
  - There is 1 variable with dimensions of time: t. This will lead to the self-similarity of the problem. That is, a solution on one time scale is a rescale of a solution on another time scale.
- Length: $\sqrt{\nu t}$
- Temperature: $\frac{E}{\sqrt{\nu t}}$

So we have

$$u^* = \frac{u}{E/(\nu t)^{n/2}}$$
$$u = \frac{E}{(\nu t)^{n/2}} u^*(\xi)$$
$$\xi = \frac{r}{\sqrt{\nu t}}$$

So our dimensionless temperature depends only on $\xi = \frac{r}{\sqrt{\nu t}}$.

Let $u^* = F$. We will look for solutions of the form

$$
\begin{aligned}
u &= \frac{E}{(\nu t)^{n/2}} F\left(\frac{r}{\sqrt{\nu t}}\right) \\
u_t &= \nu \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial u}{\partial r}\right) \\
u_t &= \frac{\left(-\frac{n}{2}\right) E}{\nu^{n/2} t^{\frac{n}{2}+1}} F + \frac{E}{(\nu t)^{n/2}} F'\left(\frac{r}{\sqrt{\nu t}}\right)\left(-\frac{1}{2}\right) \frac{r}{\sqrt{\nu} t^{3/2}} \\
u_t &= \frac{-E}{\nu^{n/2} t^{\frac{n}{2}+1}}\left[\frac{n}{2} F + \frac{1}{2} F' \frac{r}{\sqrt{\nu t}}\right] \\
&= -\frac{E}{\nu^{n/2} t^{\frac{n}{2}+1}}\left[\xi F' + nF\right] \\
\Delta u &= \frac{E}{(\nu t)^{n/2}} \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial F}{\partial r}\right) \\
&= \frac{E}{(\nu t)^{\frac{n}{2}+1}} \frac{1}{\xi^{n-1}} \frac{d}{d\xi}\left(\xi^{n-1} \frac{dF}{d\xi}\right) \\
-\frac{1}{2} \cancel{\frac{E}{\nu^{n/2} t^{\frac{n}{2}+1}}}\left[\xi F' + nF\right] &= \nu \cancel{\frac{E}{(\nu t)^{\frac{n}{2}+1}}} \frac{1}{\xi^{n-1}} \frac{d}{d\xi}\left(\xi^{n-1} \frac{dF}{d\xi}\right) \\
\frac{1}{\xi^{n-1}} \frac{d}{d\xi}\left(\xi^{n-1} \frac{dF}{d\xi}\right) &= -\frac{1}{2}(\xi F' + nF)
\end{aligned}
$$

So we have reduced our PDE to an ODE for $F(\xi)$. This is a second-order, variable coefficient ODE. We have

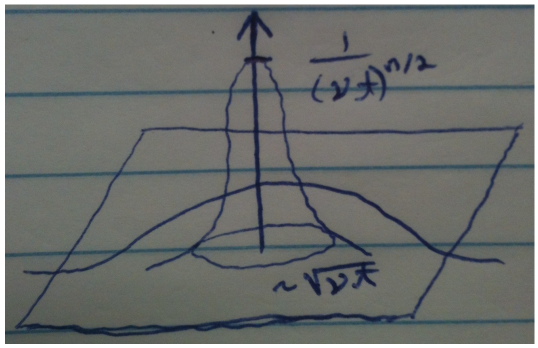
$$
\begin{aligned}
F'' + \frac{n-1}{\xi} F' &= -\frac{1}{2} \xi F' - \frac{1}{2} nF \\
F'' + \left(\frac{n-1}{\xi} + \frac{1}{2} \xi\right) F' + \frac{1}{2} nF &= 0 \\
\underbrace{\left(F' + \frac{1}{2} \xi F\right)'}_{G} + \frac{n-1}{\xi}\left(F' + \frac{1}{2} \xi F\right) &= 0 \\
\xi^{n-1} G' + (n-1) \xi^{n-2} G &= 0 \\
(\xi^{n-1} G)' &= 0 \\
G &= \frac{c}{\xi^{n-1}}
\end{aligned}
$$

Take $c = 0$; otherwise $G \to \infty$ as $\xi \to 0$ ($r \to 0$). So

$$
\begin{aligned}
G &= 0 \\
F' + \frac{1}{2} \xi F &= 0 \\
(e^{\xi^2/4} F)' &= 0 \\
e^{\xi^2/4} F &= c \qquad \text{(constant)} \\
F(\xi) &= c e^{-\xi^2/4}
\end{aligned}
$$

Using the initial condition:

$$
\begin{aligned}
\int u(x,0)\,dx &= E \\
\Rightarrow \qquad c &= \frac{1}{(4\pi)^{n/2}} \\
u(x,t) &= \frac{E}{(4\pi\nu t)^{n/2}} \exp\left(-\frac{|x|^2}{4\nu t}\right)
\end{aligned}
$$



4 4-9-12

4.1 Heat Equation

$$u_t = \nu \Delta u$$
$$u(x,0) = E\delta(x)$$

Since this is a linear PDE with constant coefficients (on $\mathbb{R}^n$), we can solve this using the Fourier transform.

4.1.1 Fourier Transform

$$\begin{aligned}
&f(x), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \\
&\hat{f}(k), \qquad k = (k_1, \ldots, k_n) \in \mathbb{R}^n \\
&\hat{f}(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-ik\cdot x}\, dx \\
&f(x) = \int_{\mathbb{R}^n} \hat{f}(k) e^{ik\cdot x}\, dk
\end{aligned}$$

We say that $\hat{f} = \mathcal{F}[f]$, where $\mathcal{F}$ is the Fourier transform. Then

$$\begin{aligned}
\frac{\partial f}{\partial x_\alpha}(x) &= \frac{\partial}{\partial x_\alpha} \int \hat{f}(k) e^{ik\cdot x}\, dk \\
&= \int \hat{f}(k) \frac{\partial}{\partial x_\alpha}(e^{ik\cdot x})\, dk \\
&= \int ik_\alpha \hat{f}(k) e^{ik\cdot x}\, dk \\
\mathcal{F}\left(\frac{\partial f}{\partial x_\alpha}\right) &= ik_\alpha \hat{f}(k)
\end{aligned}$$

In particular,

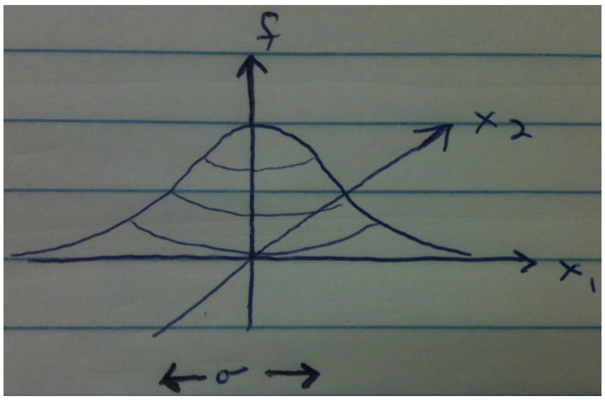
$$\mathcal{F}[\Delta f] = -|k|^2 \hat{f}(k)$$

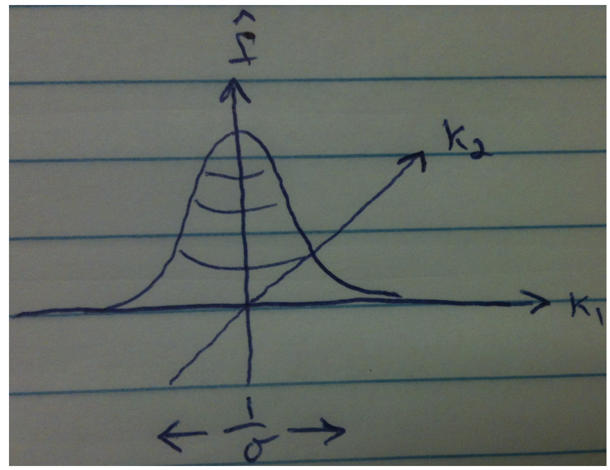
We can define $\sqrt{-\Delta}$ by

$$\mathcal{F}[\sqrt{-\Delta} f] = |k| \hat{f}(k)$$

Example 4.1.

$$f(x) = e^{-|x|^2/2\sigma^2}$$
$$\hat{f}(k) = \left(\frac{\sigma}{\sqrt{2\pi}}\right)^n e^{-\sigma^2|k|^2/2}$$





4.2 Back to the Heat Equation

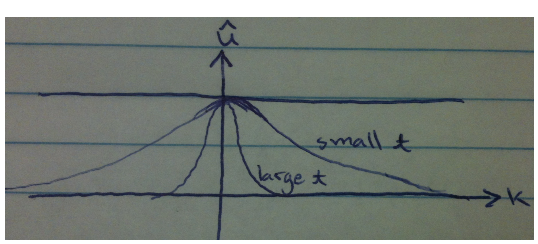
$$u(x,t) = \int_{\mathbb{R}^n} \hat{u}(k,t)e^{ik\cdot x}\,dk$$
$$\hat{u} = \mathcal{F}[u]$$
$$\mathcal{F}[u_t] = \hat{u}_t$$
$$\mathcal{F}[\Delta u] = -|k|^2\hat{u}$$
$$\mathcal{F}[\delta(x)] = \frac{1}{(2\pi)^n}\int \delta(x)e^{-ik\cdot x}\,dx = \frac{1}{(2\pi)^n}$$

So the heat equation becomes

$$\hat{u}_t = -\nu|k|^2\hat{u}$$
$$\hat{u}(k,0) = \frac{E}{(2\pi)^n}$$

The solutions look like

$$\hat{u}(k,t) = \frac{E}{(2\pi)^n}e^{-\nu|k|^2 t}$$



$$u(x,t) = \frac{E}{(4\pi\nu t)^{n/2}} e^{-|x|^2/4\nu t}$$

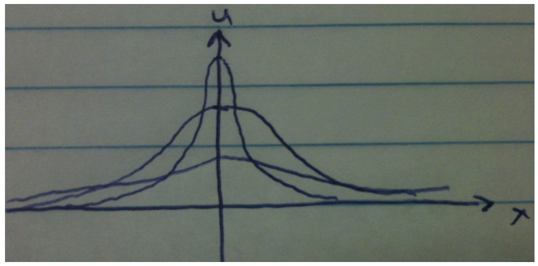


Figure 1: The heat diffuses with time.

This is a Green's function:

$$G(x,t) = \frac{1}{(4\pi\nu t)^{n/2}} e^{-|x|^2/4\nu t}.$$

$$\begin{aligned} G_t &= \nu \Delta G \\ G(x,0) &= \delta(x) \end{aligned}$$

So the solution of the heat equation,

$$\begin{aligned} u_t &= \nu \Delta u \\ u(x,0) &= f(x), \end{aligned}$$

is

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} G(x-\xi,t) f(\xi)\, d\xi \\ &= \frac{1}{(4\pi\nu t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-\xi|^2}{4\nu t}\right) f(\xi)\, d\xi. \end{aligned}$$

4.3 A Porous Medium Problem

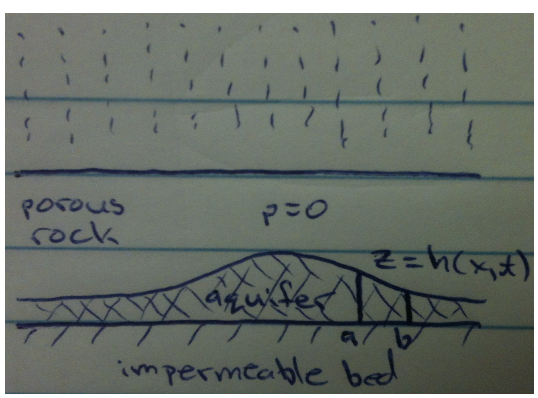


Figure 2: The aquifer is fully saturated with water. $z = h(x,t)$ is the height of the aquifer.

Assume slow transverse flow, so the pressure is hydrostatic:

$$p = \rho g(h - z).$$

The pressure head is

$$\begin{aligned} H &= p + \rho g z, \\ H &= \rho g h \qquad \text{independent of } z. \end{aligned}$$

Assume the fluid is incompressible $\Rightarrow$ conservation of volume. The change in the volume between a and b is

$$\begin{aligned} \frac{d}{dt}\int_a^b h\,dx &= -[hv]_{x=a}^{x=b} \\ &= -\int_a^b (hv)_x\,dx \\ \int_a^b [h_t + (hv)_x]\,dx = 0 & \qquad \forall\ [a,b] \\ h_t + (hv)_x = 0. & \end{aligned} \tag{4.1}$$

Darcy's law:

$$v = -\frac{k}{\mu}\nabla H.$$

k is the permeability, and μ is the fluid viscosity. This is saying that the velocity is proportional to the gradient of the pressure head. In our case, we have

$$v = -\frac{k}{\mu}\rho g h_x.$$

Plugging this into (4.1), we get

$$\begin{aligned} h_t &= K(hh_x)_x \\ K &= \frac{k\rho g}{\mu}. \end{aligned}$$

This is the 1D porous medium equation. This is a nonlinear, degenerate diffusion equation. When $h \to 0$, the diffusion drops out.

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5.1 Porous Medium Equation

$$
\begin{aligned}
h_t &= k(hh_x)_x \\
h(x,0) &= I\delta(x)
\end{aligned}
$$

(Barenblatt)

<u>Dimensions</u>

- (vertical) height H
- (horizontal) length L
- time T

<u>Dependent Variables:</u> h (H)

<u>Independent Variables:</u> x (L), t (T)

<u>Parameters:</u> k $\left(\frac{L^2}{HT}\right)$, I (HL)

Use t, k, I to nondimensionalize the problem.

$$
\begin{aligned}
[t] &= T \\
[(kIt)^{1/3}] &= L \\
\left[\frac{I}{(kIt)^{1/3}}\right] &= H \\
h(x,t) &= \frac{I^{2/3}}{(kt)^{1/3}} F\left(\frac{x}{(kIt)^{1/3}}\right) \\
\int h(x,t)\,dx &= I\int F(\xi)\,d\xi
\end{aligned}
$$

$$
-\frac{1}{3}\frac{I^{2/3}}{k^{1/3}t^{4/3}}F + \frac{I^{2/3}}{(kt)^{1/3}}\left(-\frac{1}{3}\right)\frac{x}{(kI)^{1/3}t^{4/3}}F' = k\left[\frac{I^{2/3}}{(kt)^{1/3}}\right]^2 \frac{1}{(kIt)^{2/3}}(FF')'
$$

$$
\begin{aligned}
-\frac{1}{3}F - \frac{1}{3}\xi F' &= (FF')', \qquad \xi = \frac{x}{(kIt)^{1/3}} \\
(FF')' &= -\frac{1}{3}(\xi F' + F) \\
&= -\frac{1}{3}(\xi F)' \\
FF' &= -\frac{1}{3}\xi F + c
\end{aligned}
$$

We expect $F \to 0$ as $\xi \to \infty$. Take $c = 0$.

$$
\begin{aligned}
FF' &= -\frac{1}{3}\xi F \\
F' &= -\frac{1}{3}\xi \\
F(\xi) &= \frac{1}{6}(a^2 - \xi^2)
\end{aligned}
$$

We need

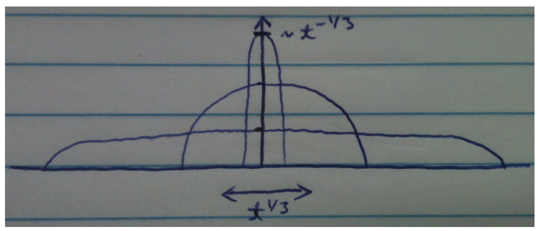
$$\int_{-\infty}^{\infty} F(\xi)\,d\xi = 1$$

$$F(\xi) = \begin{cases} \frac{1}{6}(a^2 - \xi^2) & |\xi| < a \\ 0 & |\xi| \geq a \end{cases}$$

$$\int_{-a}^{a} \frac{1}{6}(a^2 - \xi^2)\,d\xi = 1$$

$$a = \left(\frac{9}{2}\right)^{1/3}$$

$$h(x,t) = \begin{cases} \frac{I^{2/3}}{6(kt)^{1/3}}\left[\left(\frac{9}{2}\right)^{2/3} - \frac{x^2}{(kIt)^{2/3}}\right] & |x| < \left(\frac{9kIt}{2}\right)^{1/3} \\ 0 & \text{otherwise} \end{cases}$$



5.2 Perturbation Theory

$$p^{\epsilon}(x) = 0$$

Problem for x depending on a small parameter ϵ. Solution:

$$x = x(\epsilon)$$

Suppose p^{ϵ} "simplifies" at $\epsilon = 0$. Goal: to find approximations of the solution $x(\epsilon)$ when ϵ is small.

Definition 5.1. ***Regular, Singular***

Classify perturbation problem as

- *regular* if the $\epsilon = 0$ problem is "close" to the $\epsilon \neq 0$ problem
- *singular* if the $\epsilon = 0$ problem is "different" from the $\epsilon \neq 0$ problem

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6.1 Regular vs. Singular Perturbations

Example 6.1.

$$x^3 - x + \epsilon = 0$$

Look for a solutions

$$\begin{aligned} x(\epsilon) &= x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \\ x^3 &= (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^3 \\ &= x_0^3 + 3\epsilon x_0^2 x_1 + \epsilon^2[3x_0^2 x_2 + 3x_0 x_1^2] + \cdots \end{aligned}$$

$$x_0^3 + 3\epsilon x_0^2 x_1 + \epsilon^2[3x_0^2 x_2 + 3x_0 x_1^2] + \cdots - x_0 - \epsilon x_1 - \epsilon^2 x_2 - \cdots + \epsilon = 0$$

$$\begin{aligned} x_0^3 - x_0 &= 0 \\ 3x_0^2 x_1 - x_1 + 1 &= 0 \\ 3x_0^2 x_2 + -x_2 + 3x_0 x_1^2 &= 0 \end{aligned}$$

$$\begin{aligned} x_0 &= 0, \pm 1 \\ x_1 &= \frac{1}{1 - 3x_0^2} \\ x_2 &= \frac{3x_0 x_1^2}{1 - 3x_0^2} \end{aligned}$$

$$\begin{aligned} x_0 = 0: &\qquad x = 0 + \epsilon + 0 \cdot \epsilon^2 + O(\epsilon^3) \\ x_0 = 1: &\qquad x = 1 - \frac{1}{2}\epsilon - \frac{3}{8}\epsilon^2 + O(\epsilon^3) \\ x_0 = -1: &\qquad x = -1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + O(\epsilon^3) \end{aligned}$$

6.1.1 Example #2

$$\epsilon x^3 - x + 1 = 0$$
$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$
$$\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^3 - (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + 1 = 0$$
$$\epsilon x_0^3 + 3\epsilon^2 x_0^2 x_1 + \cdots - x_0 - \epsilon x_1 - \epsilon^2 x_2 + 1 = O(\epsilon^3)$$

$$-x_0 + 1 = 0$$
$$x_0^3 - x_1 = 0$$
$$3x_0^2 x_1 - x_2 = 0$$

$$x_0 = 1$$
$$x_1 = 1$$
$$x_2 = 3$$
$$x = 1 + \epsilon + 3\epsilon^2 + \cdots$$

This equation is *singular*: the cubic equation degenerates to a linear equation at $\epsilon = 0$.

We only get one root; the other two go off to ∞ as $\epsilon \to 0$. So we introduce a scaled variable:

$$x = \frac{y}{\delta(\epsilon)}, \qquad y = O(1)$$

$$\underbrace{\frac{\epsilon}{\delta^3} y^3}_{1} - \underbrace{\frac{1}{\delta} y}_{2} + \underbrace{1}_{3} = 0$$

To get a nontrivial limit, we need a dominant balance between (at least) two terms.

Two-Term Balances

- 1 $\sim$ 2: $\epsilon/\delta^3 = 1/\delta$; $\delta = \epsilon^{1/2}$; 3 ~ 1; 1, 2 $\sim 1/\epsilon^{1/2}$; 1 $\sim$ 2 $\gg$ 3
- 2 $\sim$ 3: $1/\delta = 1$; $\delta = 1$; 2, 3 ~ 1;, $1 \gg$ 1 $\sim \epsilon$
- 3 $\sim$ 1: $\epsilon/\delta^3 = 1$; $\delta = \epsilon^{1/3}$; 3, 1 ~ 1; $1 \ll$ 2 $\sim 1/\epsilon^{1/3}$

The first two are dominant balances.

To get the remaining roots... $\delta = \epsilon^{1/2}$

$$x = \frac{y}{\epsilon^{1/2}}$$
$$\frac{\epsilon}{\epsilon^{3/2}} y^3 - \frac{1}{\epsilon^{1/2}} y + 1 = 0$$
$$y^3 - y + \epsilon^{1/2} = 0$$
$$y = y_0 + \epsilon^{1/2} y_1 + \epsilon y_2 + \cdots$$

As before:

$$
\begin{aligned}
y &= 0 + \epsilon^{1/2} + O(\epsilon) \\
y &= \pm 1 - \frac{1}{2}\epsilon^{1/2} + O(\epsilon) \\
x &= 1 + \epsilon + 3\epsilon^2 + \cdots \\
x &= 1 + O(\epsilon^{1/2}) \\
x &= \pm\frac{1}{\epsilon^{1/2}} - \frac{1}{2} + O(\epsilon^{1/2})
\end{aligned}
$$

Example 6.2.

$$
\begin{aligned}
(1-\epsilon)x^2 - 2x + 1 &= 0 \\
x &= x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \\
x^2 &= x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0x_2 + x_1^2) + \cdots \\
(1-\epsilon)[x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0x_2 + x_1^2) + \cdots] - 2(x_0 + \epsilon x_1 + \epsilon^2 x_2) + 1 &= O(\epsilon^3)
\end{aligned}
$$

$$
\begin{aligned}
x_0^2 - 2x_0 + 1 &= 0 \\
2x_0x_1 - x_0^2 - 2x_1 &= 0 \\
2(x_0 - 1)x_1 &= x_0^2
\end{aligned}
$$

$$
x_0 = 1
$$

There is no solution of the assumed form (perturbing off a repeated root).

$$
x = 1 \pm \sqrt{\epsilon}
$$

The correct expansion is

$$
x = x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \cdots
$$

7 4-16-12

7.1 Asymptotic and Convergent Series

Euler 1754:

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt$$

How does $I(x)$ behave as $x \to 0^+$? This integral is well-defined for $x \geq 0$.

Formally: for small x,

$$\begin{aligned}
\frac{1}{1+xt} &= 1 - xt + (xt)^2 - \cdots + (-1)^n (xt)^n + \cdots \\
I(x) &= \int_0^\infty e^{-t}\,dt - x\int_0^\infty te^{-t}\,dt + \cdots + (-1)^n x^n \int_0^\infty t^n e^{-t}\,dt + \cdots \\
&= 1 - x + 2x^2 + \cdots + (-1)^n n! x^n + \cdots \\
I(x) &= \sum_{n=0}^\infty (-1)^n n! x^n \qquad (7.1)
\end{aligned}$$

For example, at $x = 1$:

$$\int_0^\infty \frac{e^{-t}}{1+t}\,dt = 1 - 2! + 3! - 4! + 5! \cdots$$

The ratio test shows that (7.1) has zero radius of convergence, so it diverges for all $x \neq 0$. Where did we go wrong? The expansion for $\frac{1}{1+xt}$ is only valid for $xt < 1$. So our expansion doesn't converge everywhere, namely when t is large. But when t is large, we have exponential decay in our integral.

For example, at $x = 0.1$:

$$\sum_{n=0}^{12} (-1)^n n! x^n = 0.91542$$

$$\int_0^\infty \frac{e^{-t}}{1+(0.1)t}\,dt = 0.9156$$

Theorem 7.1.

$x \geq 0,\ N = 0, 1, 2, \ldots.$

$$\left| I(x) - \sum_{n=0}^{N} (-1)^n n! x^n \right| \leq (N+1)! x^{N+1}$$

Proof.

$$\begin{aligned}
I(x) &= \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \\
&= 1 - \int_0^\infty \frac{e^{-t}}{(1+xt)^2}\,dt \\
&= 1 - x + \cdots + (-1)^N N! x^N + R_{N+1}(x) \\
R_{N+1}(x) &= (-1)^{N+1}(N+1)! x^{N+1} \int_0^\infty \frac{e^{-t}}{(1+xt)^{N+2}}\,dt \\
|R_{N+1}(x)| &\le (N+1)! x^{N+1} \underbrace{\int_0^\infty e^{-t}\,dt}_{=1}
\end{aligned}$$

□

We write this as

$$I(x) = \sum_{n=0}^{N} (-1)^n n! x^n + O(x^{N+1}) \qquad \text{as } x \to 0^+$$

$O(x^{N+1})$ stands for a term bounded by a constant times $|x|^{N+1}$.

Convergent: Fix x, $N \to \infty$
Asymptotic: Fix N, $x \to 0^+$

7.1.1 Optimal Truncation

$$\left| I(x) - \underbrace{\sum_{n=0}^{N} (-1)^n n! x^n}_{S_N(x)} \right| \le (N+1)! x^{N+1}$$

As long as the x power is beating out the factorial, the error is going down. The optimal truncation is at $N \sim \left[\frac{1}{x}\right]$. Then the error is

$$\begin{aligned}
\text{Error } &\sim \left(\frac{1}{x}\right)! x^{1/x} \\
&\sim \sqrt{\frac{2\pi}{x}} e^{-1/x} \qquad \text{as } x \to 0^+
\end{aligned}$$

where we have used Stirling's formula:

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \qquad \text{as } n \to \infty.$$

So we get exponential accuracy by optimal truncation (asymptotics beyond all orders).

7.2 Notation for Asymptotic Behavior

$f(x), g(x),\ x \to x_0\ (x_0 = 0^+, \infty, \ldots)$

We write $f(x) = O(g(x))$ as $x \to x_0$ if there exist constants $C, \delta > 0$ such that

$$|f(x)| \le C|g(x)| \qquad \text{for} \quad |x - x_0| < \delta.$$

We write that $f(x) = o(g(x))$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x)| \le \epsilon|g(x)| \qquad \text{for} \quad |x - x_0| < \delta.$$

If $g(x) \ne 0$, this is equivalent to

$$\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = 0.$$

o implies O.

Example 7.2.

$$f(x) = x$$
$$g(x) = x^2$$

As $x \to 0$, $x^2 = o(x)$. As $x \to \infty$, $x = o(x^2)$.

$$f(x) = \sin\left(\frac{1}{x}\right)$$
$$g(x) = x$$

As $x \to 0$, there is no relation between f and g. But we can say that $\sin\left(\frac{1}{x}\right) = O(1)$ as $x \to 0$.

$$f(x) = x$$
$$g(x) = 10^6 \log x$$

As $x \to \infty$, $10^6 \log x = o(x)$. Similarly, $10^6 \log(\log x) = o(\log x)$ as $x \to \infty$.

$$f(x) = x$$
$$g(x) = \log \frac{1}{x}$$

As $x \to 0$, $x = o\left(\frac{1}{\log \frac{1}{x}}\right)$.

$e^{-1/x} = o(x^n)$ as $x \to 0^+$.

8 4-18-12

8.1 Perturbation Theory for ODE's

1. Regular perturbation problems
2. Singular perturbation problems
   (a) Boundary/initial layer problems. These are treated by the method of matched asymptotic expansions (MMAE)
   (b) Oscillation problems. These are treated by the method of multiple scales (MMS)

8.2 Overdamped Simple Harmonic Oscillator (Logan 2.4)

$$\begin{aligned} m\ddot{y} + a\dot{y} + ky &= 0 \\ y(0) &= 0 \\ \dot{y}(0) &= \frac{I}{m} \end{aligned}$$

Dimensions: mass M, length L, and time T

Parameters: $m\ (M),\ a\ \left(\frac{M}{T}\right),\ k\ \left(\frac{M}{T^2}\right),\ I\ \left(\frac{ML}{T}\right)$

Variables: $y\ (L),\ t\ (T)$

For large damping, choose time scale $\frac{a}{k}$ (which has dimension T). Choose length scale $\frac{I}{a}$ (which has dimension L). Set

$$\begin{aligned} y &= \frac{I}{a}y^* \\ t &= \frac{a}{k}t^* \\ \frac{d}{dt} &= \frac{k}{a}\frac{d}{dt^*} \end{aligned}$$

(Henceforth, dots will denote derivatives with respect to t^*.) Since the equation is linear, the rescaling factor of y will cancel out. So we have

$$\begin{aligned} m\left(\frac{k}{a}\right)^2\ddot{y}^* + a\left(\frac{k}{a}\right)\dot{y}^* + ky^* &= 0 \\ y^*(0) &= 0 \\ \left(\frac{k}{a}\right)\left(\frac{I}{a}\right)\dot{y}^*(0) &= \frac{I}{m} \\ \frac{mk}{a^2}\ddot{y}^* + \dot{y}^* + y^* &= 0 \\ y^*(0) &= 0 \\ \dot{y}^*(0) &= \frac{a^2}{mk} \\ \epsilon := \frac{mk}{a^2} \end{aligned}$$

Nondimensionalized problem (drop the $*$'s):

$$\begin{aligned}
\epsilon\ddot{y} + \dot{y} + y &= 0 \\
y(0) &= 0 \\
\dot{y}(0) &= \frac{1}{\epsilon}
\end{aligned}$$

We want to find the approximate solution when ϵ is small (and positive). This is a singular perturbation problem because if we set $\epsilon = 0$ then we change the order of the ODE from 2nd order to 1st order. We can't solve a 1st order ODE with 2 initial conditions.

The solution consists of two parts:

(a) a short initial layer where $\ddot{y}$ is large $\Rightarrow$ fast

(b) long outer regions where $\ddot{y}$ is $O(1)$ $\Rightarrow$ slow

Idea: construct different "inner" and "outer" approximations, then match them.

Outer solution (b)

$$\begin{aligned}
y &= y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) \ldots \\
\epsilon\ddot{y}_0 + \epsilon^2\ddot{y}_1 + \dot{y}_0 + \epsilon\dot{y}_1 + \epsilon^2\dot{y}_2 + y_0 + \epsilon y_1 + \epsilon^2 y_2 &= O(\epsilon^3) \\
\dot{y}_0 + y_0 &= 0 \\
\ddot{y}_0\dot{y}_1 + y_1 &= 0 \\
\dot{y}_n + y_n + \ddot{y}_{n-1} &= 0 \\
y_0(t) &= ce^{-t}, \qquad t = O(1)
\end{aligned}$$

This is the leading order outer solution.

Initial layer (a)
Say $t = O(\delta)$. Introduce the time variable

$$\begin{aligned}
T &= \frac{t}{\delta} \\
\frac{d}{dt} &= \frac{1}{\delta}\frac{d}{dt} \\
y(t;\epsilon) &= Y(T;\epsilon) \\
\frac{\epsilon}{\delta^2}\frac{d^2Y}{dT^2} + \frac{1}{\delta}\frac{dY}{dT} + Y &= 0
\end{aligned}$$

The dominant balances will be

1. $\frac{1}{\delta} = 1$, $\delta = 1$ (outer)
2. $\frac{\epsilon}{\delta^2} = \frac{1}{\delta}$, $\delta = t$ (inner)
3. The third possibility, $\frac{\epsilon}{\delta^2} = 1$, is not a dominant balance

We get

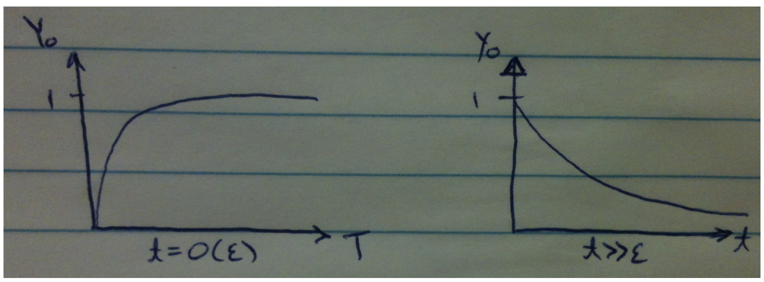
$$\begin{aligned}
\frac{d^2Y}{dT^2} + \frac{dY}{dT} + \epsilon Y &= 0 \\
Y(0) &= 0 \\
\frac{dY}{dT}(0) &= 1
\end{aligned}$$

So the inner expansion is:

$$
\begin{aligned}
Y &= Y_0(T) + \epsilon Y_1(T) + O(\epsilon^2) \\
\frac{d^2Y_0}{dT^2} + \frac{dY_0}{dT} &= 0 \\
Y_0(0) &= 0 \\
\frac{dY_0}{dT}(0) &= 1 \\
Y_0(T) &= A + Be^{-T} = 1 - e^{-T}
\end{aligned}
$$

The leading order inner solution is

$$
\begin{aligned}
Y_0(T) &= 1 - e^{-T} \\
T &= \frac{t}{\epsilon} = O(1)
\end{aligned}
$$



The matching condition is

$$
\begin{aligned}
\lim_{T\to\infty} Y_0(T) &= \lim_{t\to 0^+} y_0(t) \\
1 &= C \\
y(t,\epsilon) &\sim \begin{cases} 1 - e^{-t/\epsilon} & t = O(\epsilon) \\ e^{-t} & t = O(1) \end{cases}
\end{aligned}
$$

9 4-20-12

9.1 Strongly Damped Oscillator

Remark 9.1. ***A note on expansions***

$$\begin{aligned}
(1+x)^\alpha &= 1 + \alpha x + \frac{1}{2}\alpha(\alpha-1)x^2 + \frac{1}{3!}\alpha(\alpha-1)(\alpha-2)x^3 + \cdots, \qquad |x| < 1 \\
\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2 + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots
\end{aligned}$$

$$\begin{aligned}
\epsilon\ddot{y} + \dot{y} + y &= 0 \\
y(0) &= 0 \\
\dot{y}(0) &= \frac{1}{\epsilon}
\end{aligned}$$

The characterisitic equation, $y = e^{rt}$, gives

$$
\begin{aligned}
\epsilon r^2 + r + 1 &= 0 \\
r_{\pm} &= \frac{-1 \pm \sqrt{1-4\epsilon}}{2\epsilon} \\
r_{-} &= -\frac{1}{\epsilon} + O(1) \\
r_{+} &= \frac{-1 + \left(1 - \frac{1}{2}\cdot 4\epsilon + O(\epsilon)^2\right)}{2\epsilon} \\
&= -1 + O(\epsilon) \\
y(t) &= Ae^{r_- t} + Be^{r_+ t} \\
y(0) &= 0 \qquad A + B = 0 \\
\dot{y}(0) &= \frac{1}{\epsilon} \qquad r_- A + r_+ B = \frac{1}{\epsilon} \\
B &= -A \\
A &= \frac{1}{\epsilon}\left(\frac{1}{r_- - r_+}\right) \\
B &= \frac{1}{\epsilon}\left(\frac{1}{r_+ - r_-}\right) \\
r_+ - r_- &= \frac{-1 + \sqrt{1-4\epsilon}}{2\epsilon} - \left(\frac{-1 - \sqrt{1-4\epsilon}}{2\epsilon}\right) \\
&= \frac{\sqrt{1-4\epsilon}}{\epsilon} \\
\text{Exact solution:} \quad y(t) &= -\frac{1}{\sqrt{1-4\epsilon}} \exp\left[-\frac{\left(1+\sqrt{1-4\epsilon}\right)}{2\epsilon} t\right] \\
&\quad + \frac{1}{\sqrt{1-4\epsilon}} \exp\left[-\frac{\left(1-\sqrt{1-4\epsilon}\right)}{2\epsilon} t\right]
\end{aligned}
$$

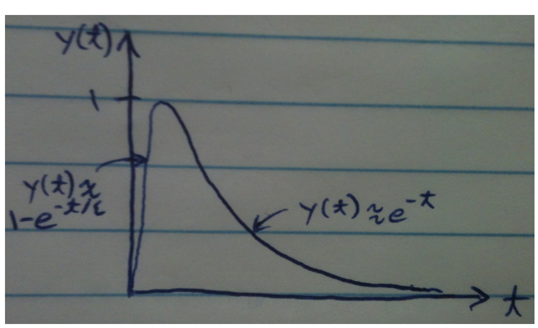
As $\epsilon \to 0^+$,

$$
\begin{aligned}
y(t) &\sim -e^{-t/\epsilon} + e^{t} \\
t &= \epsilon T \\
y &= -e^{-T} + e^{\epsilon T}
\end{aligned}
$$

Balancing $\epsilon\ddot{y} + \dot{y}$ gives $e^{-t/\epsilon}$, while balancing $\dot{y} + y$ gives e^{-t}.

As $\epsilon \to 0^+$,

$$
y(t) \sim \begin{cases} 1 - e^{-t/\epsilon} & t = O(\epsilon) \\ e^{t} & t = O(1),\ t > 0 \end{cases}
$$



9.2 Phase Plane

$$\epsilon \ddot{y} + \dot{y} + y = 0$$
$$\dot{y} = z$$
$$\dot{z} = -\frac{1}{\epsilon}(y + z)$$

Two regimes:

1. "Slow" manifold, $y + z = 0$. The approximate equation for y is then

$$\dot{y} = -y \qquad \Rightarrow \qquad y = ce^{-t}$$

2. "Fast" system, $\dot{z} = O(1/\epsilon)$ and $\dot{y} = O(1)$.

$$T = \frac{t}{\epsilon}$$
$$\frac{d}{dt} = \frac{1}{\epsilon}\frac{d}{dT}$$
$$\frac{1}{\epsilon}\frac{dy}{dT} = z$$
$$\frac{1}{\epsilon}\frac{dz}{dT} = -\frac{1}{\epsilon}(y + z)$$
$$\frac{dy}{dT} = \epsilon z \approx 0$$
$$\frac{dz}{dT} = -(y + z)$$

$y + z \neq 0$, so the approximate equation is

$$\dot{y} = 0$$
$$\dot{z} = -\frac{1}{\epsilon}(z + y)$$

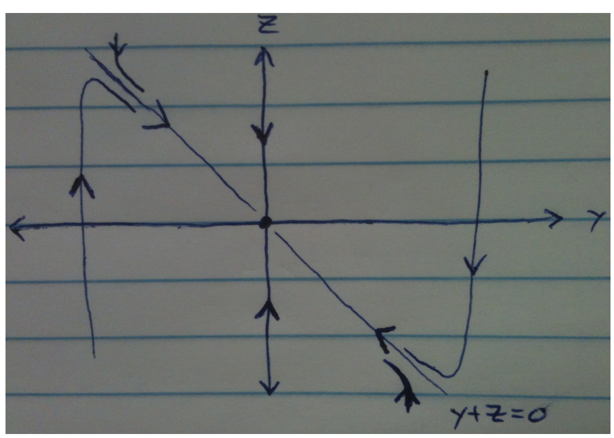


Figure 3: "Geometric Singular Perturbation Theory"

9.3 Michaelis Menton Enzyme Kinetics

$$\mathrm{H_2O_2} \rightarrow \mathrm{H_2O} + \mathrm{O}$$
$$\mathrm{E} + \mathrm{S} \overset{k_0}{\longleftarrow} \overset{k_1}{\longrightarrow} \mathrm{C} \overset{k_2}{\longrightarrow} \mathrm{P}$$

Law of mass actions:

$$\text{rate of reaction} \propto \text{product of concentrations},$$

where the constant of proportionality is the rate constant.

- $e(t) =$ concentration of E
- $s(t) =$ concentration of S
- $c(t) =$ concentration of C
- $p(t) =$ concentration of P

$$\begin{aligned}
\frac{de}{dt} &= -k_1 e_s + (k_0 + k_2)c \\
\frac{ds}{dt} &= -k_1 es + k_0 c \\
\frac{dc}{dt} &= k_1 es - (k_0 + k_2)c \\
\frac{dp}{dt} &= k_2 c
\end{aligned}$$

We see that

$$\begin{aligned}
\frac{d}{dt}(e + c) &= 0 \\
e + c &= \text{constant}
\end{aligned}$$

10 4-23-12

10.1 Enzyme Kinetics (Continued)

$$\mathrm{E} + \mathrm{S} \overset{k_0}{\longleftarrow} \overset{k_1}{\longrightarrow} \mathrm{C} \overset{k_2}{\longrightarrow} \mathrm{P}$$

$$\begin{aligned}
\frac{de}{dt} &= -k_1 e_s + (k_0 + k_2)c \\
\frac{ds}{dt} &= -k_1 es + k_0 c \\
\frac{dc}{dt} &= k_1 es - (k_0 + k_2)c \\
\frac{dp}{dt} &= k_2 c \\
e(0) &= e_0 \\
s(0) &= s_0 \\
c(0) &= 0 \\
p(0) &= 0 \\
e + c &= e_0 \\
\frac{d}{dt}[e + c] &= 0 \\
\frac{de}{dt} &= -k_1 es + (k_0 + k_2)(e_0 - e) \\
\frac{ds}{dt} &= -k_1 es + k_0(e_0 - e)
\end{aligned}$$

Dimensions: time T, concentration C

Independent Variables: $t\ (T)$

Dependent Variables: $e\ (C),\ s\ (C)$

Parameters: $e_0\ (C),\ s_0\ (C),\ k_0\ \left(\frac{1}{T}\right),\ k_1\ \left(\frac{1}{CT}\right),\ k_2\ \left(\frac{1}{T}\right)$

$$
\begin{aligned}
u(\tau) &= \frac{s(t)}{s_0} \\
v(\tau) &= \frac{c(t)}{e_0} \\
\tau &= k_1 e_0 t
\end{aligned}
$$

$$
\begin{aligned}
\frac{du}{d\tau} &= -u + (u + k - \lambda)v \\
\epsilon\frac{dv}{d\tau} &= u - (u + k)v \\
u(0) &= 1 \\
v(0) &= 0 \\
\epsilon &= \frac{e_0}{s_0} \\
k &= \frac{k_0 + k_2}{k_1 s_0} \\
\lambda &= \frac{k_2}{k_1 s_0}
\end{aligned}
$$

We have two regimes:

(a) Short time, $\tau = O(\epsilon)$

(b) Long time, $\tau = O(1)$

(b) **Long time.** Expand

$$
\begin{aligned}
u &= u_0(\tau) + \epsilon u_1(\tau) + \cdots \\
v &= v_0(\tau) + \epsilon v_1(\tau) + \cdots \\
\frac{du_0}{d\tau} &= -u_0 + (u_0 + k - \lambda)v_0 \\
0 &= u_0 - (u_0 + k)v_0 \\
v_0 &= \frac{u_0}{u_0 + k} \\
\frac{du_0}{d\tau} &= -u_0 + (u_0 + k - \lambda) \cdot \frac{u_0}{u_0 + k} \\
&= -\frac{\lambda u_0}{u_0 + k}
\end{aligned}
$$

(a) **Short time.**

$$
\begin{aligned}
T &= \frac{\tau}{\epsilon} \\
\frac{d}{dt} &= \frac{1}{\epsilon}\frac{d}{dT} \\
U(T) &= u(t) \\
\frac{dU}{dT} &= \epsilon[-U + (U + k - \lambda)V] \\
\frac{dV}{dT} &= U - (U + k)V \\
U &= U_0 + \epsilon U_1 + \cdots \\
V &= V_0 + \epsilon V_1 + \cdots \\
\frac{dU_0}{dT} &= 0 \\
\frac{dV_0}{dT} &= U_0 - (U_0 + k)V_0 \\
U_0(0) &= 1 \\
V_0(0) &= 0 \\
U_0(T) &= 1 \\
\frac{dV_0}{dT} &= 1 - (1 + k)V_0 \\
V_0(0) &= 0 \\
V_0(T) &= \frac{1 - e^{-(1+k)T}}{1 + k}
\end{aligned}
$$

(b) **Matching.**

$$u_0(0) = \lim_{T\to\infty} U_0(T) = 1$$

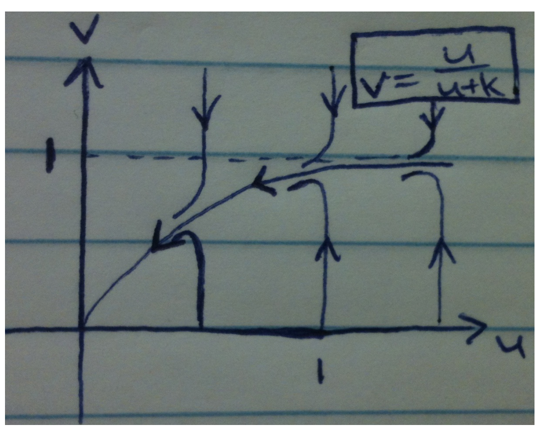


Figure 4: $\mathrm{E} + \mathrm{S} \overset{k_0}{\longleftarrow} \overset{k_1}{\longrightarrow} \mathrm{C} \overset{k_2}{\longrightarrow} \mathrm{P}$

11 4-25-12

11.1 Geometric Singular Perturbation Theory

$$\epsilon \dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

$x(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^n,\ f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m,\ g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. x contains the "fast" variables, y contains the "slow" variables. Introduce a fast time: $T = \frac{t}{\epsilon}$. Let $' = \frac{d}{dT}$ and $\dot{} = \frac{d}{dt}$. So $\frac{1}{\epsilon}\frac{d}{dT} = \frac{d}{dt}$.

$$x' = f(x,y)$$
$$y' = \epsilon g(x,y)$$

"Slow" system:

$$f(x,y) = 0$$
$$\dot{y} = g(x,y)$$

"Fast" system:

$$x' = f(x,y)$$
$$y' = 0$$

The slow manifold is $f(x,y) = 0$. We can't satisfy all of the initial data in the slow system, because the initial data for x has to be such that $f(x,y) = 0$. Physicists say that the x variable is a slave to the y variable.

For the fast system, $y = y_0$ (constant) and $x' = f(x, y_0)$.

Simplest case:

- The slow manifold is a graph, $x = \phi(y),\ \phi : \mathbb{R}^n \to \mathbb{R}^m$.

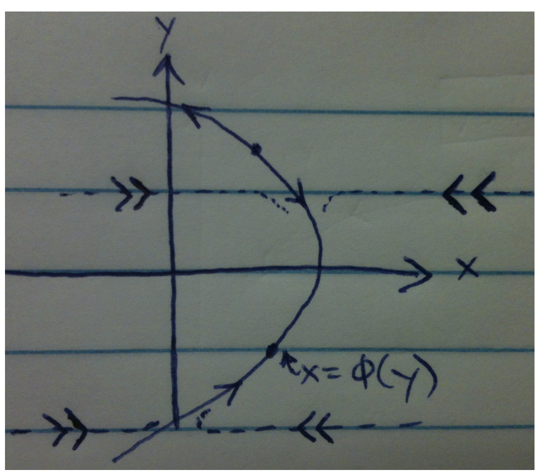
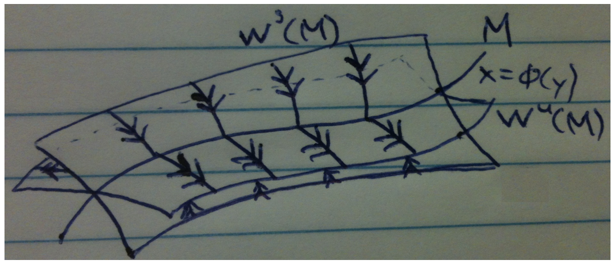


Figure 5: $f(\phi(y), y) = 0,\ \dot{y} = g(\phi(y), y)$.

- Assume that $x = \phi(y)$ is a globally asymptotically stable (unique) equilibrium for the "fast" equation, $x' = f(x,y)$.

Tikhonov (1948) and Levinson (1949) gave a theory for attracting slow manifolds in these "fast-slow" systems.

Fenichel (1971) proved that the full system has an invariant manifold close to the slow manifold for small ϵ provided $x = \phi(y)$ is a hyperbolic equilibrium of the "fast" system $x' = f(x, y)$.



11.2 Van der Pol Oscillator

$$\underbrace{\epsilon\ddot{x} + (x^2 - 1)\dot{x}}_{=-\dot{y}} + x = 0$$

Small mass/large damping: $0 < \epsilon \ll 1$
Negative damping/excitability: $|x| < 1$
Positive damping: $|x| > 1$

Lienard variables:

$$y = x - \frac{1}{3}\dot{x}^3 - \epsilon\dot{x}$$

$$\epsilon\dot{x} = x - \frac{1}{3}x^3 - y$$
$$\dot{y} = x$$

Slow manifold: $y = x - \frac{1}{3}x^3$

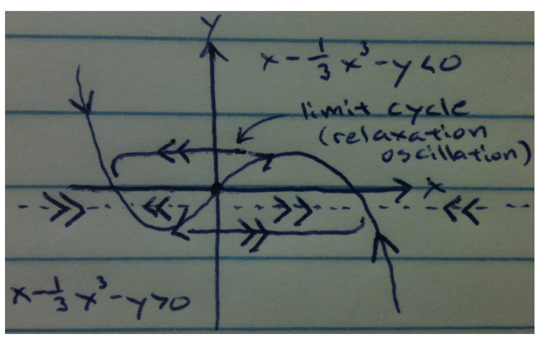


Figure 6:

Slow system

$$y = x - \frac{1}{3}x^3$$
$$\dot{y} = x$$

Fast system

$$x' = x - \frac{1}{3}x^3 - y$$
$$y' = 0$$

12 4-27-12

12.1 Heat Flow in a Slowly-Varying Rod

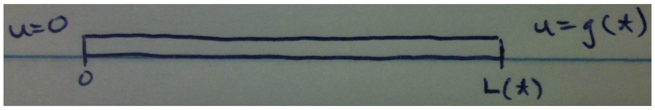
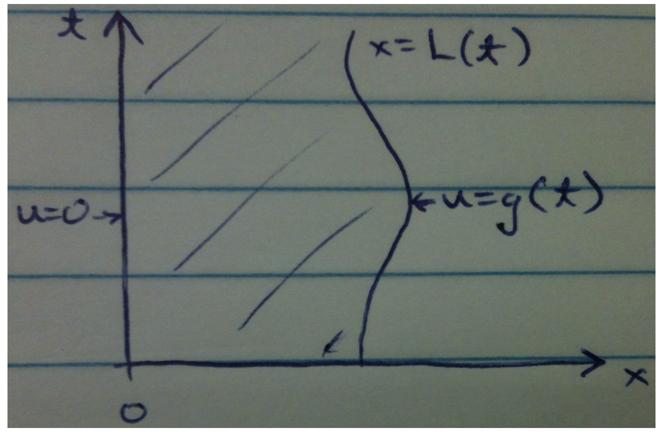


Figure 7: $u(x,t) = \text{temperature}$



$$
\begin{aligned}
u_t &= \nu u_{xx}, \qquad 0 < x < L(t),\ t > 0 \\
u(0,t) &= 0 \\
u(L(t),t) &= g(t) \\
u(x,0) &= f(x)
\end{aligned}
$$

Nondimensionalization

$$\begin{aligned}
L_0 &= L(0) \\
T_0 &= \text{ time-scale of variations in } L(t) \\
\theta &= \text{ typical temperature} \\
L(t) &= L_0 L^* \left(\frac{t}{T_0}\right) \\
g(t) &= \theta_0 g^* \left(\frac{t}{T_0}\right) \\
f(x) &= \theta_0 f^* \left(\frac{x}{L_0}\right) \\
x^* &= \frac{x}{L_0} \\
t^* &= \frac{t}{T_0} \\
u^* &= \frac{u}{\theta_0} \\
\partial_x &= \frac{1}{L_0}\partial_{x^*} \\
\partial_t &= \frac{1}{T_0}\partial_{t^*} \\
u_t &= \frac{\theta_0}{T_0}u^*_{t^*} \\
u_{xx} &= \frac{\theta_0}{L_0^2}u^*_{x^*x^*} \\
u_t &= \nu u_{xx} \\
\frac{\theta_0}{T_0}u^*_{t^*} &= \frac{\nu\theta_0}{L_0^2}u^*_{x^*x^*} \\
\epsilon u^*_{t^*} &= u^*_{x^*x^*} \\
\epsilon &= \frac{L_0^2}{\nu T_0}
\end{aligned}$$

So we have

$$\begin{aligned}
\epsilon u^*_{t^*} &= u^*_{x^*x^*}, \qquad 0 < x^* < L^*(t^*),\ t^* > 0 \\
u^*(0, t^*) &= 0 \\
u^*(L^*(t^*), t^*) &= g^*(t^*) \\
u^*(x^*, 0) &- f^*(x^*)
\end{aligned}$$

Interpretation of ϵ:

- T_d = diffusion-timescale, i.e. time, for heat to diffuse from one end of the rod to the other. $L \sim \sqrt{\nu T} \iff T \sim L^2/\nu$.
- $T_d = \frac{L_0^2}{\nu}$
- $\epsilon = \frac{T_d}{T_0}$

Assume $\epsilon \ll 1$. This means that heat diffuses rapidly over the rod relative to the timescale of variations in the length/boundary data.

Drop the *'s.

$$
\begin{aligned}
\epsilon u_t &= u_{xx}, \qquad 0 < x < L(t),\ t > 0 \\
u(0,t) &= 0 \\
u(L(t),t) &= g(t) \\
u(x,0) &= f(x), \qquad 0 < x < 1,\ L(0) = 1
\end{aligned}
$$

<u>Outer expansion:</u>

$$
\begin{aligned}
u &= u_0(x,t) + \epsilon u_1(x,t) + O(\epsilon^2) \\
u_{0,xx} &= 0, \qquad 0 < x < L \\
u_0(0,t) &= 0 \\
u_0(L,t) &= g
\end{aligned}
$$

We have to drop the initial condition (because we wouldn't be able to satisfy it with the outer solution).

$$
\begin{aligned}
u_0(x,t) &= A(t)x + B(t) \\
&= \frac{g(t)}{L(t)} x
\end{aligned}
$$

<u>Inner expansion:</u>

$$
\begin{aligned}
T &= \frac{t}{\epsilon} \\
u(x,t;\epsilon) &= U(x,T;\epsilon) \\
\partial_t &= \frac{1}{\epsilon}\partial_T
\end{aligned}
$$

$$
\begin{aligned}
U_t &= U_{xx}, \qquad 0 < x < L(\epsilon T),\ T > 0 \\
U(0,T) &= 0 \\
U(L(\epsilon T), \epsilon T) &= g(\epsilon T) \\
U(x,0) &= f(x), \qquad 0 < x < 1
\end{aligned}
$$

$$
U = U_0(x,T) + \epsilon U_1(x,T) + O(\epsilon^2)
$$

$$
\begin{aligned}
U_{0,T} &= U_{0,xx}, \qquad 0 < x < 1, T > 0 \\
U_0(0,T) &= 0 \\
U_0(1,T) &= g(0) \\
U_0(x,0) &= f(x), \qquad 0 < x < 1
\end{aligned}
$$

Solve by separating variables.

$$U(x,T) = g(0)X + V(x,T)$$

$$\begin{aligned} V_t &= V_{xx} \\ V(0,T) &= 0 \\ V(1,T) &= 0 \\ V(x,0) &= f(x) - g(0)x \end{aligned}$$

$$\begin{aligned} V(x,T) &= \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 T} \sin(n\pi x) \\ c_n &= 2\int_0^1 [f(x) - g(0)x] \sin(n\pi x)\, dx \\ U_0(x,T) &= g(0)x + V(x,T) \end{aligned}$$

So we have

$$\begin{aligned} \text{Outer solution:} \quad u_0(x,t) &= \frac{g(t)}{L(t)}x \\ \text{Inner solution:} \quad U_0(x,T) &= g(0)x + V(x,T) \end{aligned}$$

Do they match?

$$\begin{aligned} \lim_{T\to\infty} U_0(x,T) &= g(0)x \\ \lim_{t\to 0^+} u_0(x,t) &= g(0)x \end{aligned}$$

Uniform solution:

$$\begin{aligned} u &\sim u_{\text{inner}} + u_{\text{outer}} - u_{\text{matching}} \\ &\sim \frac{g(t)}{L(t)}x + V\left(x, \frac{t}{\epsilon}\right) \end{aligned}$$

13 4-30-12

13.1 Boundary Layer Problems

Navier-Stokes equation for incompressible fluid:

$$\vec{u}_t \vec{u} \cdot \nabla \vec{u} + \nabla p = \epsilon \Delta \vec{u}, \qquad \epsilon = \frac{1}{\mathrm{Re}}$$
$$\nabla \cdot \vec{u} = 0 \qquad (\text{“no slip” condition})$$
$$\vec{u}(\vec{x}, 0) = \vec{u}_0(\vec{x})$$
$$\vec{u}(\vec{x}, t) = 0 \qquad \text{on } \partial\Omega$$

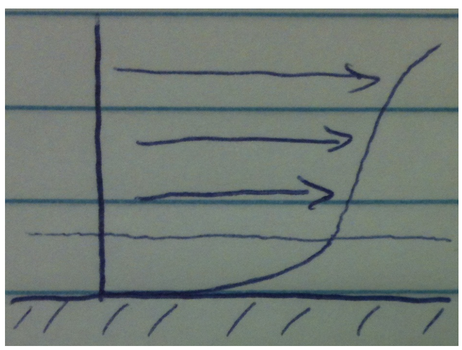
Setting $\epsilon = 0$ (no viscosity), we get the Euler equation:

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \nabla p = 0$$

The Euler equation with no-slip boundary condition is overdetermined. So we impose the “no-flow” condition:

$$\vec{u} \cdot \vec{n} = 0$$

Prandtl (1905) introduced boundary layer theory.



The velocity goes quickly from zero to something large, so the derivative is very large.

13.2 Model Boundary Layer Problem

$$\epsilon y'' + 2y' + y = 0, \qquad 0 < x < 1$$
$$y(0) = 0$$
$$y(1) = 1$$

We want to find an asymptotic approximation of the solution for $0 < \epsilon \ll 1$.

Straightforward (outer) expansion:

$$y = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + O(\epsilon^3)$$
$$2y_0' + y_0 = 0$$
$$2y_1' + y_1 + y_0'' = 0$$
$$2y_n' + y_n + y_{n-1}'' = 0$$

Problem: can't satisfy both BC's because the order of the ODE drops from 2 to 1 at $\epsilon = 0$. It turns out that the correct BC to impose is the BC at $x = 1$.

$$\begin{aligned} y_0(1) &= 1 \\ y_1(1) &= 0 \\ y_n(1) &= 0 \end{aligned}$$

$$\begin{aligned} y_0(x) &= ce^{-x/2} \\ &= e^{1/2}e^{-x/2} \end{aligned}$$

So we get a boundary layer near $x = 0$ where the solution adjusts rapidly from $\approx e^{1/2}$ to 0 at $x = 0$.

<u>Inner expansion (near $x = 0$):</u>

$$\begin{aligned} X &= \frac{x}{\delta} \\ y(x;\epsilon) &= Y(X;\epsilon) \\ y'(x;\epsilon) &= \frac{1}{\delta}\frac{dY}{dX} = \frac{1}{\delta}Y' \end{aligned}$$

$$\underbrace{\frac{\epsilon}{\delta^2}}_{1} + \underbrace{\frac{2}{\delta}Y'}_{2} + \underbrace{Y}_{3} = 0$$

Dominant balances:

- $1 \sim 2$: $\frac{\epsilon}{\delta^2} = \frac{1}{\delta} \;\Rightarrow\; \delta = \epsilon$, $3 \ll 1 \sim 2$
- $2 \sim 3$: $\delta = 1 \;\Rightarrow\; 1 \ll 2 \sim 3$
- $1 \sim 3$: $\frac{\epsilon}{\delta^2} = 1 \;\Rightarrow\; \delta = \epsilon^{1/2}$, $2 \gg 1 \sim 3$

Take $\delta = \epsilon$.

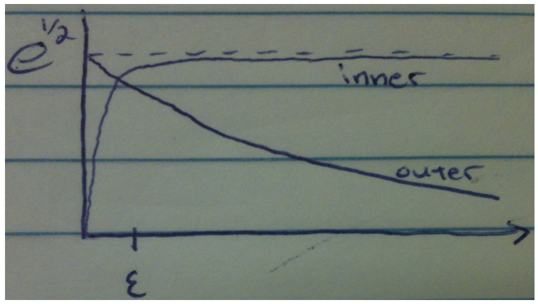
$$\begin{aligned} Y'' + 2Y' + \epsilon Y &= 0 \\ Y &= Y_0(X) + \epsilon Y_1(X) + \cdots \\ Y_0'' + 2Y_0' &= 0 \\ Y_1'' + 2Y_1' + Y_0 &= 0 \\ Y_0(0) &= 0 \\ Y_0' &= ce^{-2X} \\ Y_0(X) &= c_1 + c_2e^{-2X} = c(1 - e^{-2X}) \end{aligned}$$

Matching condition:

$$\begin{aligned} \lim_{x\to 0^+} y_0(x) &= \lim_{X\to\infty} Y_0(X) \\ e^{1/2} &= c \\ Y_0(X) &= e^{1/2}(1 - e^{-2X}) \end{aligned}$$

Leading-order asymptotic solution:

$$y(x;\epsilon) \sim \begin{cases} e^{1/2}e^{-x/2} & \text{as } \epsilon \to 0^+,\; 0 < x \le 1 \\ e^{1/2}(1 - e^{-2x/\epsilon}) & 0 \le \frac{x}{\epsilon} < \infty \end{cases}$$



Uniform solution:

$$y_{\text{inner}} + y_{\text{outer}} - y_{\text{overlap}}$$
$$y(x;\epsilon) \sim e^{1/2}(e^{-x/2} - e^{-2x/\epsilon})$$

Let's compare this to the exact solution. The characteristic equation is

$$\epsilon r^2 + 2r + 1 = 0$$
$$r = \frac{-1 \pm \sqrt{1-\epsilon}}{\epsilon}$$
$$r = -\alpha(\epsilon),\ -\frac{\beta(\epsilon)}{\epsilon}$$
$$\beta(\epsilon) = 2 + \cdots$$
$$-1 + \sqrt{1-\epsilon} = -1 + \left(1 - \frac{1}{2}\epsilon\right) = -\frac{1}{2}\epsilon$$
$$y(x;\epsilon) = \frac{e^{-\alpha x} - e^{-\beta x/\epsilon}}{e^{-\alpha} - e^{-\beta/\epsilon}}$$
$$\sim \frac{e^{-x/2} - e^{-2x/\epsilon}}{e^{-1/2} - e^{-2/\epsilon}}$$

This agrees with the uniform solution (to leading order in ϵ).

14 5-2-12

14.1 Follow-Up: Why is the boundary layer at 0?

$$\begin{aligned} \epsilon y'' + 2y' + y &= 0, \qquad 0 < x < 1 \\ y(0) &= 0 \\ y(1) &= 1 \end{aligned}$$

Try to find the solution with the boundary layer at $x = 1$.

(a) Outer solution.

$$y = y_0 + \epsilon y_1(x) + \cdots$$

$$\begin{aligned} 2y_0' + y_0 &= 0, \qquad 0 < x < 1 \\ y_0(0) &= 0 \end{aligned}$$

$$y_0 = ce^{-x/2} \quad \Rightarrow \quad y_0 = 0$$

(b) Inner solution near $x = 1$.

$$\begin{aligned} X &= \frac{1-x}{\epsilon} \\ y(x;\epsilon) &= Y(X;\epsilon) \\ \frac{d}{dx} &= -\frac{1}{\epsilon}\frac{d}{dX} \\ Y'' - 2Y' + \epsilon Y &= 0, \qquad 0 < X < \infty \qquad \left(Y' = \frac{dY}{dX}\right) \\ Y(0) &= 1 \end{aligned}$$

$$\begin{aligned} Y &= Y_0 + \epsilon Y_1 + \cdots \\ Y_0'' - 2Y_0' &= 0 \\ Y_0(0) &= 1 \\ Y_0(X) &= c_1 + c_2 e^{2X} \\ &= 1 + c(1 - e^{2x}) \end{aligned}$$

(c) Matching. We want $y_0(x)$ as $x \to 1^-$ to match with $Y_0(X)$ as $X \to \infty$.

$$\begin{aligned} y_0(x) &\to 0 \quad \text{as } x \to 1^- \\ Y_0(x) &\to \begin{cases} \infty & c > 0 \\ 1 & c = 0 \\ -\infty & c < 0 \end{cases} \end{aligned}$$

So after going through all of this analysis, we find that it won't work.

14.2 General Linear 2nd Order BVP's

$$\begin{aligned} \epsilon y'' + a(x)y' + b(x)y &= 0, \qquad 0 < x < 1 \\ y(0) &= \alpha \\ y(1) &= \beta \end{aligned}$$

Find an asymptotic solution as $\epsilon \to 0^+$. Suppose $a(x) \geq \delta > 0$ on $0 \leq x \leq 1$.

Claim: we get a boundary layer at $x = 0$.

1. $X = \frac{x}{\epsilon}$. The leading order inner equation for Y_0 is

$$\begin{aligned} Y_0'' + a(0)Y_0' &= 0 \\ Y_0(X) &= c_1 + c_2 e^{-a(0)X} \\ &\to c_1 \quad \text{as } X \to \infty \text{ if } a(0) > 0 \end{aligned}$$

2. $X = \frac{1-x}{\epsilon}$ for a boundary layer at $x = 1$.

$$\begin{aligned} Y_0'' - a(1)Y_0' &= 0 \\ Y_0(X) &= c_1 + c_2 e^{a(1)X} \end{aligned}$$

   We need $a(1) < 0$ in order to permit matching.

So

1. If $a(x) \geq \delta > 0$ we get a boundary layer at $x = 0$.
2. If $a(x) \leq -\delta < 0$ we get a boundary layer at $x = 1$

If $a(x)$ changes sign (*turning points*), we get more complicated behavior.

3. If $a(0) < 0,\ a(1) > 0$, we get no boundary layers (maybe interior/corner layer).
4. If $a(0) > 0,\ a(1) < 0$, we can have boundary layers at both endpoints.

14.2.1 Boundary Layer Example 1

$$\begin{aligned} \epsilon y'' + xy' - y &= 0, \qquad -1 < x < 1 \\ y(-1) &= 1 \\ y(1) &= 2 \end{aligned}$$

$$\left\{ \begin{array}{l} a(-1) = -1 < 0 \\ a(1) = 1 > 0 \end{array} \right. \quad \Rightarrow \text{ no BL possible at either endpoint}$$

(a) Outer solution.

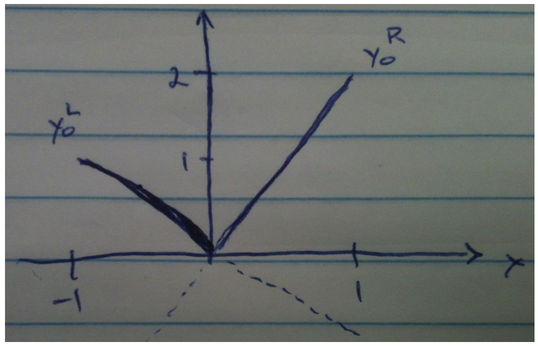
$$\begin{aligned} y &= y_0(x) + \epsilon y_1(x) + \cdots \\ xy_0' - y_0 &= 0 \\ y_0(x) &= Cx \end{aligned}$$

Impose left and right boundary conditions to get left and right outer solutions.

$$\begin{aligned} y_0^L(x) &= -x \\ y_0^R(x) &= 2x \end{aligned}$$

Try

$$y_0(x) = \left\{ \begin{array}{ll} -x & -1 \leq x < 0 \\ 2x & 0 < x \leq 1 \end{array} \right.$$



(b) Inner solution. Introduce scaled variable

$$X = \frac{x}{\delta}$$
$$y(x) = \delta Y(X)$$
$$\frac{d}{dx} = \frac{1}{\delta}\frac{d}{dX}$$
$$x = \delta X$$

$$\frac{\epsilon}{\delta^2}Y'' + \delta X \cdot \frac{1}{\delta}Y' - Y = 0$$
$$\frac{\epsilon}{\delta^2}Y'' + XY' - Y = 0$$

We have a dominant three-term balance for $\delta = \epsilon^{1/2}$.

$$Y'' + XY' - Y = 0, \qquad -\infty < X < \infty$$

Matching.

$$y_0^L(x) = -\delta\left(\frac{x}{\delta}\right) = -\delta X$$
$$y_0^R(x) = \delta\left(\frac{2x}{\delta}\right) = \delta 2X$$
$$Y(X) \sim -X \quad \text{as } X \to -\infty$$
$$Y(x) \sim 2X \quad \text{as } X \to \infty$$

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15.1 Boundary Layers (Continued)

$$
\begin{aligned}
\epsilon y'' + a(x)y' + b(x)y &= 0 \\
y(0) &= \alpha \\
y(1) &= \beta
\end{aligned}
$$

A boundary layer at $x = 0$ is possible if $a(0) > 0$, and a boundary layer at $x = 1$ is possible if $a(1) < 0$. If $a(x)$ changes signs, more complications may occur.

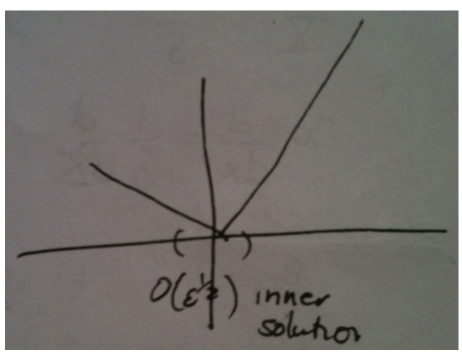
15.1.1 Boundary Layer Example 1 (From Last Time)

$$
\begin{aligned}
\epsilon y'' + xy' - y &= 0, \qquad -1 < x < 1 \\
y(-1) &= 1 \\
y(1) &= 2
\end{aligned}
$$

There was no way to put in a boundary layer at either endpoint because as x changes signs you change from growing to decaying solutions.

<u>Outer solution:</u>

$$
\begin{aligned}
y &= y_0(x) + \epsilon y_1(x) + \cdots \\
xy_0' - y_0 &= 0 \\
y_0(x) &= Cx \\
y_0^L(x) &= -x \\
y_0^R(x) &= 2x
\end{aligned}
$$



The simplest, where we have a corner layer at $x = 0$, is the right solution because it can be matched.

<u>Inner solution:</u> (for the corner layer)

$$
\begin{aligned}
y &= \epsilon^{1/2} Y\left(\frac{x}{\epsilon^{1/2}}\right) \\
X &= \frac{x}{\epsilon^{1/2}}
\end{aligned}
$$

Here we have a 3-term dominant balance, and we get

$$
Y_0'' + xY_0' - Y_0 = 0
$$

and then we have to subject this to the matching conditions.

Matching conditions:

$$\text{inner limit of outer solution} = \text{outer limit of inner solution}$$

$$\begin{aligned} y_0^L(x) &= -x \\ &= -\epsilon^{1/2}\frac{x}{\epsilon^{1/2}} \\ &= -\epsilon^{1/2}X \end{aligned} \qquad\qquad \begin{aligned} y_0^R(x) &= 2x \\ &= 2\epsilon^{1/2}X \end{aligned}$$

The solution

$$Y_0(X) = c_1 X + c_2\left[e^{-\frac{1}{2}X} + X\int_{-\infty}^{x} e^{-t^2/2}\,dt\right]$$

as $X \to -\infty$, and this looks like $c_1 X$, so let $c_1 = -1$. As $X \to \infty$,

$$\begin{aligned} Y_0(X) &\sim \left[c_1 + c_2\int_{-\infty}^{\infty} e^{-t^2/2}\,dt\right]x \\ c_2 &= \frac{3}{\sqrt{2\pi}} \end{aligned}$$

Question: what is the uniform solution? It would look like

$$\begin{aligned} y &\sim y_{\text{inner}} + y^L_{\text{outer}} + y^R_{\text{outer}} - y^L_{\text{overlap}} - y^R_{\text{overlap}} \\ y &\sim -x + \frac{3\epsilon^{1/2}}{\sqrt{2\pi}}e^{-x^2/2\epsilon} + \frac{3}{\sqrt{2\pi}}x\int_{-\infty}^{x/\epsilon^{1/2}} e^{-t^2/2}\,dt \end{aligned}$$

More important than using the inner solution is that it matches with respect to the boundaries and outer solution.

15.1.2 Boundary Layer Example 2

$$\begin{aligned} \epsilon y'' - xy' + y &= 0, \qquad -1 < x < 1 \\ y(-1) &= 1 \\ y(1) &= 2 \end{aligned}$$

So here $a(x) = -x,\ a(-1) = 1$, and $a(1) = -1$ (so boundary layers are possible at both $x = -1$ and $x = 1$).

Outer solution: (away from any boundary layers)

$$\begin{aligned} y &= y_0(x) + \epsilon y_1(x) + \cdots \\ -xy_0' + y_0 &= 0 \\ y_0(x) &= cx \end{aligned}$$

We'll leave c arbitrary since it is not clear which BC to impose.

Inner solution at $x = -1$:

$$\begin{aligned}
X &= \frac{x+1}{\epsilon} \\
y(x;\epsilon) &= Y(X;\epsilon) \\
\frac{d}{dx} &= \frac{1}{\epsilon}\frac{d}{dX} \\
x &= -1 + \epsilon X \\
\frac{1}{\epsilon}Y'' - (-1+\epsilon X)\frac{1}{\epsilon}Y' + Y &= 0 \\
Y(0;\epsilon) &= 1 \\
Y &= Y_0(X) + \epsilon Y_1(X) + \cdots \\
Y_0'' + Y_0' &= 0 \\
Y_0(0) &= 1 \\
Y_0(X) &= 1 + A(1 - e^{-X})
\end{aligned}$$

Matching condition at $x = 1$:

$$\begin{aligned}
\lim_{X\to\infty} Y(X) &= \lim_{x\to -1} y_0(x) \\
1 + A &= -c
\end{aligned}$$

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16.1 Boundary Layer Example 2

$$\begin{aligned} \epsilon y'' - xy' + y &= 0, \qquad -1 < x < 1 \\ y(-1) &= 1 \\ y(1) &= 2 \end{aligned}$$

Boundary layers are possible at both endpoints.

Outer expansion:

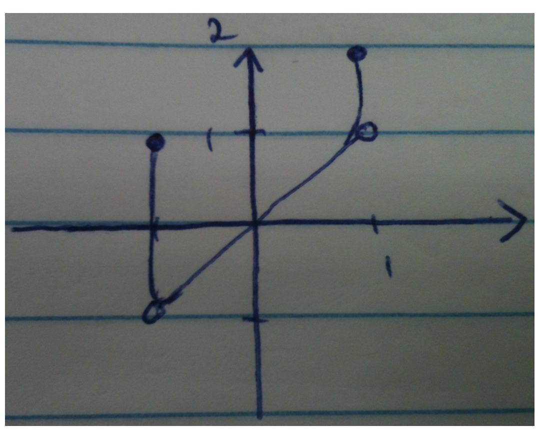
$$\begin{aligned} y &= y_0(x) + \epsilon y_1(x) + \cdots \\ -xy_0' + y_0 &= 0 \\ y_0(x) &= Cx \end{aligned}$$

Inner expansion ($x = -1$):

$$\begin{aligned} X &= \frac{x+1}{\epsilon} \quad \left(= \frac{x-1}{\delta} \right) \\ Y(X;\epsilon) &= y(x;\epsilon) \\ Y &= Y_0(X) + \epsilon Y_1(X) + \cdots \\ Y_0'' + Y_0' &= 0 \\ Y_0(X) &= 1 + A(1 - e^{-X}) \qquad \big(Y_0(0) = 1\big) \end{aligned}$$

Matching at $x = -1$:

$$\begin{aligned} \lim_{x \to -1^+} y_0(x) &= \lim_{X \to \infty} Y_0(X) \\ -C &= 1 + A \end{aligned}$$



Inner expansion ($x = 1$):

$$\begin{aligned}
X &= \frac{1-x}{\epsilon} \\
Y(X;\epsilon) &= y(x;\epsilon) \\
\frac{d}{dx} &= -\frac{1}{\epsilon}\frac{d}{dX} \\
\frac{1}{\epsilon}Y'' + \frac{1}{\epsilon}(1+\epsilon X)Y' + Y &= 0, \qquad Y(0;\epsilon) = 2 \\
Y &= Y_0(X) + \epsilon Y_1(X) + \cdots \\
Y_0'' + Y_0' &= 0, \qquad Y_0(0) = 2 \\
Y_0(X) &= 2 + B(1 - e^{-X})
\end{aligned}$$

Matching:

$$\begin{aligned}
\lim_{x\to 1} y_0(x) &= \lim_{X\to\infty} Y_0(X) \\
C &= 2 + B
\end{aligned}$$

So the solution is

$$y \sim \begin{cases} -1 + A\left[1 - e^{-(1+x)/\epsilon}\right] \\ Cx \\ 2 + B\left[1 - e^{-(1-x)/\epsilon}\right] \end{cases}$$

$$\begin{aligned}
-C &= 1 + A \\
C &= 2 + B
\end{aligned}$$

The problem is that C is undetermined. It remains undetermined to all orders in ϵ.

We can find C here by using symmetry of the problem.

$$\begin{aligned}
y(x) &= \frac{1}{2}x + z(x) \\
\epsilon z'' - x\left(\frac{1}{2} + z'\right) + \frac{1}{2}x + z &= 0 \\
\epsilon z'' - xz' + z &= 0 \\
z(-1) &= \frac{3}{2} \\
z(1) &= \frac{3}{2}
\end{aligned}$$

This is invariant under $x \to -x,\ z \to z$. So for a solution $y = \frac{1}{2}x + z$ (assuming it's unique), z is an even function of x.

$$y \sim \begin{cases} -C - Ae^{-(1+x)/\epsilon} \\ Cx \\ C - Be^{-(1-x)/\epsilon} \end{cases}$$

$$\begin{aligned}
-C &= 1 + A \\
C &= 2 + B \\
C &= \frac{1}{2} \\
A = B &= -\frac{3}{2}
\end{aligned}$$

This holds in the leading order solution if $C = \frac{1}{2}$, which implies that $A = B = -\frac{3}{2}$.

$$y(x) \sim \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-(1+x)/\epsilon} & 1 + x = O(\epsilon) \\ \frac{1}{2}x & -1 < x < 1 \\ \frac{1}{2} + \frac{3}{2}e^{-(1-x)/\epsilon} & 1 - x = O(\epsilon) \end{cases}$$

The uniform solution would be

$$\begin{aligned} y_{\text{uniform}} &\sim -\frac{1}{2} + \frac{3}{2}e^{-(1+x)/\epsilon} + \frac{1}{2}x + \frac{1}{2} + \frac{3}{2}e^{-(1-x)/\epsilon} - \left(-\frac{1}{2}\right) - \frac{1}{2} \\ &= \frac{1}{2}x + \frac{3}{2}\left[e^{-(1+x)/\epsilon} + e^{-(1-x)/\epsilon}\right] \end{aligned}$$

16.2 Boundary Layer Example 3

$$\begin{aligned} \epsilon y'' - yy' + y = 0, &\qquad 0 < x < 1 \\ y(0) = 1 & \\ y(1) = -1 & \end{aligned}$$

A comparison with the linear equation suggests no boundary layer at $x = 0$ or $x = 1$.

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17.1 Boundary Layer Example 3

$$\begin{aligned} \epsilon y'' - yy' + y = 0, \qquad 0 < x < 1 \\ y(0) = 1 \\ y(1) = -1 \end{aligned}$$

Look for a solution with no boundary layers at $x = 0$ or $x = 1$.

Outer solution:

$$\begin{aligned} y &= y_0(x) + \epsilon y_1(x) + \cdots \\ -y_0 y_0' + y_0 &= 0 \\ y_0(-y_0' + 1) &= 0 \end{aligned}$$

Either

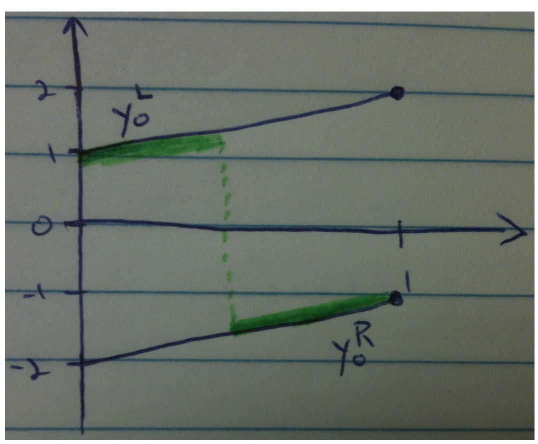
$$\begin{aligned} y_0 &= 0 \\ y_0' &= 1, \qquad y_0 = x + c \end{aligned}$$

The left outer solution is

$$\begin{aligned} y_0^L(x) &= x + 1 \\ y_0^L(0) &= 1 \end{aligned}$$

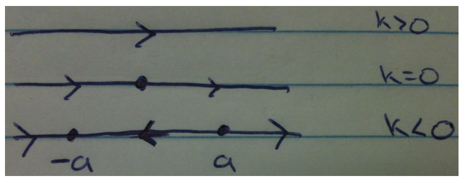
The right outer solution is

$$\begin{aligned} y_0^R(x) &= x - 2 \\ y_0^R(1) &= -1 \end{aligned}$$



Look for an interior layer of width $O(\epsilon)$ where, at x_0 $(0 < x_0 < 1)$, the solution jumps from the left outer solution to the right outer solution.

$$
\begin{aligned}
X &= \frac{x - x_0}{\epsilon} \\
Y(X;\epsilon) &= y(x;\epsilon) \\
\frac{d}{dx} &= \frac{1}{\epsilon}\frac{d}{dX} \\
Y'' - YY' + \epsilon Y &= 0 \\
Y &= Y_0(X) + \epsilon Y_1(X) + \cdots \\
Y_0'' - Y_0 Y_0' &= 0 \\
Y_0' - \frac{1}{2}Y_0^2 &= k \\
Y_0' &= k + \frac{1}{2}Y_0^2
\end{aligned}
$$



Matching:

$$
\begin{aligned}
k &= -\frac{1}{2}a^2 < 0 \qquad (a > 0) \\
Y_0' &= -\frac{1}{2}a^2 + \frac{1}{2}Y_0^2 \\
Y_0(X) &\to a \qquad \text{as } X \to -\infty \\
Y_0(X) &\to -a \qquad \text{as } X \to \infty
\end{aligned}
$$

This requires that $x_0 = \frac{1}{2}$ in order to jump from $-a$ to a.

Matching condition:

$$
\begin{aligned}
\lim_{X\to\infty} Y_0(X) &= \lim_{x\to x_0^+} y_0^R(x) \qquad -a = -\frac{3}{2} \\
\lim_{X\to-\infty} Y_0(X) &= \lim_{x\to x_0^-} y_0^L(x) \qquad a = \frac{3}{2}
\end{aligned}
$$

So $a = \frac{3}{2}$. The solution is

$$
Y_0(x) = -\frac{3}{2}\tanh\left[\frac{3}{4}(X - c)\right]
$$

This constant c is left undetermined (to all orders in ϵ). Note that the system is invariant under $x \to 1 - x$, $y \to -y$ (and the boundary conditions also remain unchanged). So the solution (if unique) must be odd about $x = \frac{1}{2}$. So $y\left(\frac{1}{2}\right) = 0$ and therefore $c = 0$.

<u>Summary:</u>

$$
y \sim \begin{cases} x + 1 & 0 \le x < \frac{1}{2} \\ -\frac{3}{2}\tan\left[\frac{3\left(x - \frac{1}{2}\right)}{4\epsilon}\right] & x - \frac{1}{2} = O(\epsilon) \\ x - 2 & \frac{1}{2} < x \le 1 \end{cases}
$$

The uniform (composite) solution is

$$y(x) \sim x - \frac{1}{2} - \frac{3}{2}\tan\left[\frac{3\left(x-\frac{1}{2}\right)}{4\epsilon}\right]$$

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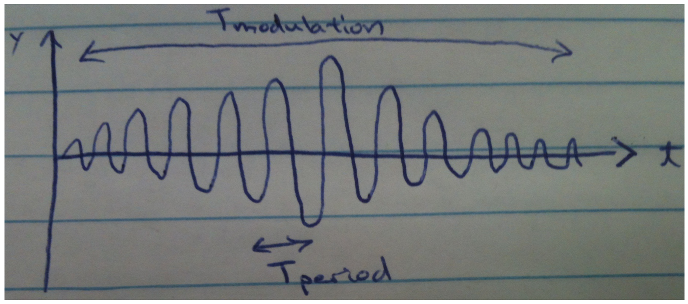
18.1 Method of Multiple Scales (MMS) and Oscillations



Pendulum

$$\ddot{x} + \sin x = 0$$

Linearized equation at $x = 0$:

$$\begin{aligned}
\ddot{x} + x &= 0 \qquad \text{(simple harmonic oscillator)} \\
x(t) &= a\cos t + b\sin t \\
&= Ae^{it} + A^*e^{-it}, \\
A &= \frac{a - ib}{2}
\end{aligned}$$

Look for small-amplitude solutions of the nonlinear equation (weakly nonlinear). Introduce a small parameter $\epsilon > 0$ and look for solutions

$$x(t,\epsilon) = \epsilon x_1(t) + \epsilon^3 x_2(t) + \epsilon^5 x_3(t) + O(\epsilon^7)$$

For example, we could have

$$\begin{aligned}
x(0,\epsilon) &= \epsilon \\
\dot{x}(0,\epsilon) &= 0 \\
\sin x &= x - \frac{1}{6}x^3 + O(x^5) \\
&= \epsilon x_1 + \epsilon^3 x_2 - \frac{1}{6}\epsilon^3 x_1^3 + O(\epsilon^5) \\
\epsilon\ddot{x}_1 + \epsilon^3\ddot{x}_2 + \epsilon x_1 + \epsilon^3\left(x_2 - \frac{1}{6}x_1^3\right) + O(\epsilon^5) &= 0 \\
O(\epsilon): \qquad \ddot{x}_1 + x_1 &= 0 \\
O(\epsilon^3): \qquad \ddot{x}_2 + x_2 &= \frac{1}{6}x_1^3 \\
x_1(t) &= Ae^{it} + A^*e^{-it} \\
&= Ae^{it} + \underbrace{\text{c.c.}}_{\text{complex conjugate}} \\
\ddot{x}_2 + x_2 &= \frac{1}{6}\left[Ae^{it} + A^*e^{-it}\right]^3 \\
&= \frac{1}{6}\left[A^3e^{3it} + 3|A|^2Ae^{it} + 3|A|^2A^*e^{-it} + (A^*)^3e^{-3it}\right]
\end{aligned}$$

Side calculation: the solution of

$$\begin{aligned}\ddot{y} + y &= Ce^{3it} \\ y(t) &= De^{3it} \\ \ddot{y} + y &= (-9+1)De^{3it} \\ &= -8De^{3it} \\ D &= -\frac{1}{8}C\end{aligned}$$

Another side calculation: consider

$$\ddot{y} + y = Ce^{it}$$

e^{it} is a solution of the homogeneous equation, so try

$$\begin{aligned}y(t) &= Dte^{it} \\ \dot{y} &= D(it+1)e^{it} \\ \ddot{y} &= D(-t+i)e^{it} + iDe^{it} \\ &= D(-t+2i)e^{it} \\ \ddot{y} + y &= 2iDe^{it} \\ D &= \frac{C}{2i}\end{aligned}$$

Back to our problem, we have

$$x_2(t) = -\frac{A^3}{48}e^{3it} + \frac{|A|^2 A}{4i}te^{it} - \frac{|A|^2 A^*}{4i}te^{it} - \frac{(A^*)^3}{48}e^{-3it} + Be^{it} + B^* e^{-it}$$

Note: terms like te^{it} appear in $x_2(t)$. The actual solution is a periodic function of time! Terms like te^{it} are called *secular terms*.

The perturbation expansion becomes invalid when $t = O\left(1/\epsilon^2\right)$ and $\epsilon^2 x_2 = O(\epsilon x_1)$.

18.1.1 Example

The origin of secular terms is the change in period/frequency of nonlinear oscillations with amplitude:

$$\begin{aligned}\epsilon\cos((1+\epsilon^2)t) &= \epsilon\cos(t+\epsilon^2 t) \\ &= \epsilon\cos t - (\sin t)\epsilon^3 t + O(\epsilon^4)\end{aligned}$$

There is a nonuniformity in the expansion as $\epsilon \to 0$ for large t. In a sense, the largeness of t overcomes the smallness of ϵ.

18.2 Poincaré-Lindstedt Method

Introduce a rescaled time,

$$\tau = \omega(\epsilon)t.$$

Expand the frequency as

$$\omega(\epsilon) = 1 + \epsilon^2\omega_2 + \cdots .$$

Choose ω_2 to ensure that no secular terms appear.

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19.1 Poincaré-Lindstedt Method

Pendulum:

$$\ddot{x} + \sin x = 0$$

We want to obtain an asymptotic solution for small-amplitude periodic solutions. Straightforward expansion fails due to secular terms (from dependence of the period on amplitude).

Idea: introduce a "strained" time

$$\tau = \omega t$$
$$x(t) = y(\omega t) = y(\tau)$$

Recall that $y(\tau)$ is 2π-periodic in τ. The 2π is for convenience. The important point is that the period of $y(\tau)$ is fixed.

$$\frac{d}{dt} = \omega \frac{d}{d\tau}$$
$$\dot{x} = \omega \dot{y}, \qquad \dot{y} = \frac{dy}{d\tau}$$
$$\omega^2 \ddot{y} + \sin y = 0$$

Expand:

$$\begin{aligned}
y &= \epsilon y_1(\tau) + \epsilon^3 y_2(\tau) + \cdots \\
\omega &= \omega_0 + \epsilon^2 \omega_1 + \cdots \\
y(\tau + 2\pi) &= y(\tau) \\
\sin y &= y - \frac{1}{6} y^3 + O(y^5) \\
&= \epsilon y_1 + \epsilon^3 y_2 - \frac{1}{6}\epsilon^3 y_1^3 + O(\epsilon^5) \\
2\epsilon^2 \omega_0 \omega_1 &\leftarrow \epsilon \omega_0^2 \ddot{y}_1 + \epsilon^3 \left[\omega_0^2 \ddot{y}_2 + 2\omega_0 \omega_1 \ddot{y}_1\right] + \cdots
\end{aligned}$$

$$(\omega_0^2 + 2\epsilon^2 \omega_0 \omega_1 + \cdots)(\epsilon \ddot{y}_1 + \epsilon^3 \ddot{y}_2 + \cdots) + \epsilon y_1 + \epsilon^3 \left(y_2 - \frac{1}{6} y_1^3\right) = O(\epsilon^5)$$

$$\begin{aligned}
O(\epsilon): \qquad \omega_0^2 \ddot{y}_1 + y_1 &= 0 \\
y_1(\tau + 2\pi) &= y_1(\tau) \\
O(\epsilon^3): \qquad \omega_0^2 \ddot{y}_2 + y_2 &= \frac{1}{6} y_1^3 - 2\omega_0 \omega_1 \ddot{y}_1 \\
y_2(\tau + 2\pi) &= y_2(\tau)
\end{aligned}$$

From the leading order equation, we need $\omega_0^2 = 1$ ($\omega_0 = 1$). Then

$$y_1(\tau) = A e^{i\tau} + A^* e^{-i\tau}$$

Next order:

$$\begin{aligned}
\ddot{y}_2 + y_2 &= \frac{1}{6} y_1^3 - 2\omega_1 \ddot{y}_1 \\
y_2(\tau + 2\pi) &= y_2(\tau) \\
\ddot{y}_2 + y_2 &= \frac{1}{6}\left(A^3 e^{3i\tau} + 3A^2 A^* e^{i\tau} + 3a(A^*)^2 e^{-i\tau} + (A^*)^3 e^{-3i\tau}\right) + 2\omega_1 (A e^{i\tau} + A^* e^{-i\tau}) \\
&= \frac{1}{6} A^3 e^{3i\tau} + \left[\frac{1}{2} A|A|^2 + 2\omega_1 A\right] e^{it} + \left[\frac{1}{2} A^* |A|^2 + 2\omega_1 A^*\right] e^{-i\tau} + \frac{1}{6}(A^*)^3 e^{-3i\tau}
\end{aligned}$$

The solution has the form

$$y_2(\tau) = Be^{3i\tau} + C\tau e^{i\tau} + \text{ complex conjugates}$$

$C\tau e^{i\tau}$ is a secular term (non-periodic), from the resonant term $\propto e^{i\tau}$ that solution of the homogeneous equation. We only get a periodic solution for $y_2(\tau)$ if the coefficient of $e^{i\tau}$ on the RHS is zero. So

$$\begin{aligned}
\frac{1}{2}|A|^2 + 2\omega_1 &= 0 \\
\omega_1 &= -\frac{1}{4}|A|^2 \\
\ddot{y}_2 + y_2 &= \frac{1}{6}A^3 e^{3i\tau} + \text{ complex conjugates} \\
y_2(\tau) &= Be^{3i\tau} + \text{ complex conjugates} \\
-9B + B &= \frac{1}{6}A^3 \\
B &= -\frac{1}{48}A^3 \\
y(\tau) &= Ae^{i\tau} + \text{ complex conjugate } - \frac{1}{48}\epsilon^3 A^3 e^{3i\tau} + \text{ complex conjugate } + O(\epsilon^3) \\
\omega &= 1 - \frac{1}{4}\epsilon^2 |A|^2 + O(\epsilon^4) \\
x(t;\epsilon) &= y(\omega t; \epsilon) \\
&= \epsilon A e^{i\omega t} - \frac{1}{48}\epsilon^3 A^3 e^{3i\omega t} + \text{ complex conjugate } + O(\epsilon^5) \\
\omega(\epsilon) &= 1 - \frac{1}{4}\epsilon^2 |A|^2 + O(\epsilon^4)
\end{aligned}$$

For example, consider the solution with

$$\left.\begin{aligned} x &= a \\ \dot{x} &= 0 \end{aligned}\right\} \quad \text{at } t = 0$$

$$\begin{aligned}
\epsilon(A + A^*) - \frac{1}{48}\epsilon^3[A^3 + (A^*)^3] &= a + \cdots \\
i\omega\epsilon(A - A^*) + \frac{1}{48}\cdot 3i\omega\epsilon^3[A^3 - (A^*)^3] &= 0 + \cdots \\
A &= A^* \quad \text{is real} \\
2\epsilon A - \frac{1}{24}\epsilon^3 A^3 &= a \\
\epsilon A &= \frac{1}{2}a + O(\epsilon^3) \\
&= \frac{1}{2}a + \frac{1}{384}a^3 + O(a^5)
\end{aligned}$$

So we are solving

$$\begin{aligned}
\ddot{x} + \sin x &= 0 \\
x(0) &= a \\
\dot{x}(0) &= 0 \\
x(t) &= \frac{1}{2}ae^{i\omega t} + \frac{1}{2}ae^{-i\omega t} + \frac{1}{384}a^3(e^{i\omega t} + e^{-i\omega t}) - \frac{1}{384}a^3(e^{3i\omega t} + e^{-3i\omega t}) + O(a^5) \\
x(t) &= a\cos(\omega t) + \frac{1}{192}a^3[\cos(\omega t) - \cos(3\omega t)] + O(a^5) \\
\omega &= 1 - \frac{1}{16}a^2 + O(a^4)
\end{aligned}$$

The period of the solution is

$$T = \frac{2\pi}{\omega} = 2\pi \left(\frac{1}{1 - \frac{1}{16}a^2 + \cdots} \right)$$
$$= 2\pi \left(1 + \frac{1}{16}a^2 + O(a^4) \right)$$

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20.1 Poincaré-Lindstedt Method

$$\ddot{x} + x = \epsilon F(t, x, \dot{x})$$

Look for periodic solutions.

$$\begin{aligned}
\tau &= \omega t \\
\omega^2 \frac{d^2 x}{d\tau^2} + x &= \epsilon F\left(t, x, \omega \frac{dx}{d\tau}\right) \\
x(\tau + 2\pi; \epsilon) &= x(\tau; \epsilon) \\
x(\tau; \epsilon) &= x_0(\tau) + \epsilon x_1(\tau) + \cdots \\
\omega &= \omega_0 + \epsilon \omega_1 + \cdots \\
\omega_0^2 \frac{d^2 x_0}{d\tau^2} + x_0 &= 0 \\
\omega_0 &= 1 \qquad \text{to get } 2\pi\text{-periodic solutions} \\
x_0 &= A e^{i\tau} + A^* e^{-i\tau} \\
\frac{d^2 x_n}{d\tau^2} + x_n &= f_n, \qquad f_n \text{ depends on } x_0, \ldots, x_{n-1} \text{ and } \omega_1, \ldots, \omega_{n-1}
\end{aligned}$$

This has the form

$$\begin{aligned}
L x_n &= f_n \\
L &= \frac{d^2}{d\tau^2} + 1 \qquad \text{acting on } 2\pi\text{-periodic functions } x_n \in L^2(\mathbb{T})
\end{aligned}$$

L is a self-adjoint (Sturm-Liouville) operator with periodic BC's.

$$\begin{aligned}
\langle f, g \rangle &= \int_0^{2\pi} \overline{f(\tau)} g(\tau) \, d\tau \\
\langle f, Lg \rangle &= \langle Lf, g \rangle
\end{aligned}$$

The eigenvalues are

$$L\phi = \lambda \phi$$

$$\begin{aligned}
\lambda_0 &= 1 & \phi_0 &= 1 \\
\lambda_n &= -n^2 + 1 & \phi_n &= e^{int}, e^{-int}
\end{aligned}$$

For $f \in L^2(\mathbb{T})$, when is $Lu = f$ solvable? If $L\phi = 0$,

$$\begin{aligned}
\langle \phi, Lu \rangle &= \langle \phi, f \rangle \\
\langle L\phi, u \rangle &= \langle \phi, f \rangle \\
\langle \phi, f \rangle &= 0
\end{aligned}$$

Fredholm alternative: $Lu = f$, $L^* = L$ is solvable only if

$$\langle \phi, f \rangle = 0 \qquad \forall\ \phi \text{ such that } L\phi = 0.$$

(The eigenfunction expansion shows it is sufficient also.)

For $L = \frac{d^2}{d\tau^2} + 1$,

$$
\begin{aligned}
L\phi &= 0 \\
\phi &= c_1 e^{i\tau} + c_2 e^{-i\tau}
\end{aligned}
$$

The solvability condition is

$$\left\langle e^{i\tau}, f \right\rangle = \left\langle e^{-i\tau}, f \right\rangle = 0$$

which says that the Fourier coefficients $\hat{f}_1$ and $\hat{f}_{-1}$ vanish.

$$
\begin{aligned}
Lx_0 &= 0 \\
x_0 &= Ae^{i\tau} + A^* e^{-i\tau} \\
Lx_n &= f_n(x_0, \ldots, x_{n-1}, \omega_1, \ldots, \omega_{n-1}) \\
x_n &= x_n^{(p)} + A_n e^{i\tau} + A_n^* e^{-i\tau}
\end{aligned}
$$

Determine ω_{n-1} and (possibly) $|A_{n-1}|$ from the solvability conditions for x_n.

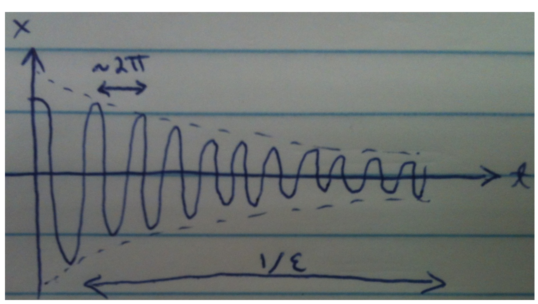
20.2 Weakly Damped Simple Harmonic Oscillator

$$\ddot{x} + \epsilon \dot{x} + x = 0, \qquad 0 < \epsilon \ll 1$$

Straightforward expansion:

$$
\begin{aligned}
x &= x_0(t) + \epsilon x_1(t) + \cdots \\
\ddot{x}_0 + x_0 &= 0 \\
x_0 &= Ae^{it} + A^* e^{-it} \\
\ddot{x}_1 + x_1 &= -\dot{x}_0 \\
\ddot{x}_1 + x_1 &= -iAe^{it} + iA^* e^{-it}
\end{aligned}
$$

Get te^{-it} terms in x_1 (secular). Here, introducing a variable $\tau = \omega t$ and looking for periodic solutions in τ doesn't help!



The solutions look like e^{rt}.

$$
\begin{aligned}
r^2 + \epsilon r + 1 &= 0 \\
r &= -\frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2} \\
&= -\frac{\epsilon}{2} \pm i\sqrt{1 - \frac{\epsilon^2}{4}}
\end{aligned}
$$

Basic idea: we have two time-scales

1. The period of oscillations, $O(1) \;\Rightarrow\; t = t$
2. The time-scale of the damping, $O\left(\frac{1}{\epsilon}\right) \;\Rightarrow\; T = \epsilon t$

Introduce two "multiple-scale" variables simultaneously. Look for solutions of the form

$$x = x(t, T; \epsilon)$$

and treat t and T as independent variables. (Evaluate $T = \epsilon t$ at the end.) This *seems* crazy because we have replaced an ODE with a PDE.

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21.1 Weakly Damped Oscillator

$$\text{(ODE)} \qquad \ddot{x} + \epsilon\dot{x} + x = 0$$

We want to obtain an asymptotic solution that is valid for long times, $t = O\left(\frac{1}{\epsilon}\right)$. Straightforward expansion for $x(t;\epsilon)$ leads to secular terms. For the method of multiple scales, we will introduce two time variables: $t,\ T = \epsilon t$. Look for a solution of the form

$$x(t;\epsilon) = y(t, \epsilon t; \epsilon).$$

Then

$$\begin{aligned}
\dot{x}(t;\epsilon) &= y_t(t,\epsilon t;\epsilon) + \epsilon y_T(t,\epsilon t;\epsilon)\\
\ddot{x} &= y_{tt} + 2\epsilon y_{tT} + \epsilon^2 y_{TT}\\
\frac{d}{dt} &\to \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T} \qquad \text{(derivative expansion)}
\end{aligned}$$

$$\text{(PDE)} \qquad y_{tt} + 2\epsilon y_{tT} + \epsilon^2 y_{TT} + \epsilon(y_t + \epsilon y_T) + y = 0$$

$x(t;\epsilon)$ satisfies the ODE if and only if $y(t,T;\epsilon)$ satisfies the PDE on $T = \epsilon t$. The idea of the method of multiple scales is to require that $y(t,T;\epsilon)$ satisfies the PDE for all (t,T). So we start by introducing a lot of freedom, requiring that $x(t;\epsilon) = y(t,\epsilon t;\epsilon)$, and then we take it away by saying that it must satisfy the PDE for all (t,T).

Expand:

$$\begin{aligned}
y(t,T;\epsilon) &= y_0(t,T) + \epsilon y_1(t,T) + O(\epsilon^2)\\
O(1): \qquad y_{0,tt} + y_0 &= 0\\
O(\epsilon): \qquad y_{1,tt} + y_1 + 2y_{0,tT} + y_{0,t} &= 0\\
y_0(t,T) &= A(T)e^{it} + A^*(T)e^{-it}\\
y_{1,tt} + y_1 + 2iA_Te^{it} + \text{ complex conjugate } + iAe^{it} + \text{ complex conjugate } &= 0\\
y_{1,tt} + y_1 + i(2A_T + A)e^{it} - i(2A_T^* + A^*)e^{-it} &= 0\\
y_1(t,T) &= Cte^{it}\\
y_{1,tt} + y_1 &= C(-te^{it} + 2e^{it}) + Cte^{it} = 2iCe^{it}\\
C &= -\left(A_T + \frac{1}{2}A\right)\\
y_1(t,T) &= -\left(A_T + \frac{1}{2}A\right)te^{it} + \text{ complex conjugate}\\
&\quad + Be^{it} + \text{ complex conjugate}
\end{aligned}$$

We require that the $y_n(t,T)$ don't grow too fast in t (e.g. bounded functions of t or sublinear). We get that

$y_1(t,T)$ is a bounded (periodic) function of t only if the coefficient of e^{it} vanishes:

$$
\begin{aligned}
2A_T + A &= 0 \\
A(T) &= A_0 e^{-T/2} \\
y_0(t,T) &= A_0 e^{-T/2} e^{it} + A_0^* e^{-T/2} e^{-it} \\
x(t;\epsilon) &= A_0 e^{-\epsilon t/2} e^{it} + \text{ complex conjugate } + O(\epsilon) \qquad \text{for } t = O\left(\frac{1}{\epsilon}\right)
\end{aligned}
$$

$$
\begin{aligned}
r^2 + \epsilon r + 1 &= 0 \\
r &= -\frac{\epsilon}{2} \pm i\sqrt{1 - \frac{1}{4}\epsilon^2}
\end{aligned}
$$

21.2 van der Pol Oscillator

We already looked at strong damping:

$$\epsilon \ddot{x} + (x^2 - 1)\dot{x} + x = 0.$$

Weak damping:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0.$$

Strong damping:

$$
\begin{aligned}
\dot{x} &= y \\
\epsilon \dot{y} &= x - (x^2 - 1)y \\
\text{Slow manifold:} \qquad y &= \frac{x}{1 - x^2}
\end{aligned}
$$

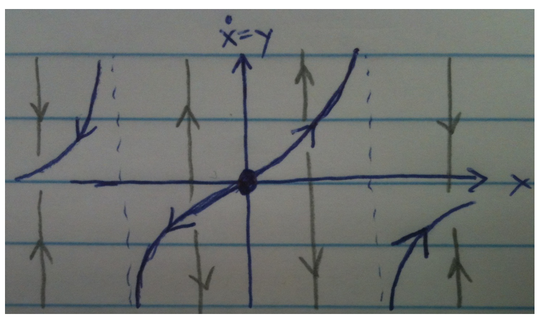

Figure 8: There is a limit cycle in here somewhere. This is why we use the Lienard variables... (See Figure 6.)

Weak damping:

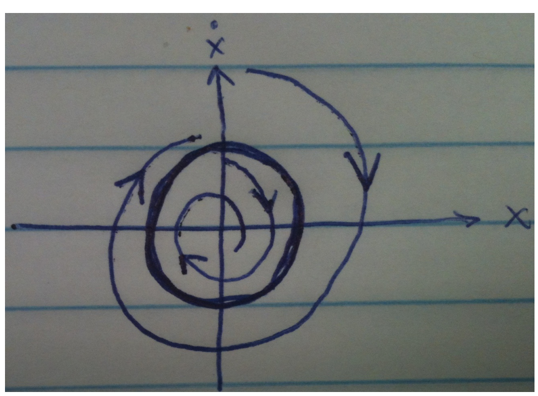

Figure 9: We spiral into the limit cycle from the outside, and we spiral away from the limit cycle on the inside.

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22.1 van der Pol Equation

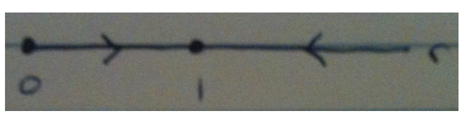
$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \qquad \text{(weak damping)}$$

Multiple scale variables: $t,\ T = \epsilon t$. Look for a solution of the form

$$\begin{aligned}
x(t;\epsilon) &= y(t, \epsilon t; \epsilon) \\
\frac{d}{dt} &\to \left.\frac{\partial}{\partial t}\right|_T + \epsilon \left.\frac{\partial}{\partial T}\right|_t \\
y_{tt} + 2\epsilon y_{tT} + \epsilon^2 y_{TT} + \epsilon(y^2 - 1)(y_t + \epsilon y_T) + y &= 0 \\
y_{tt} + \epsilon\left[2y_{tT} + (y^2 - 1)y_t\right] + \epsilon^2\left[y_{TT} + (y^2 - 1)y_T\right] + y &= 0 \\
y &= y_0(t, T) + \epsilon y_1(t, T) + O(\epsilon^2) \\
y_{0,tt} + y_0 &= 0 \\
y_{1,tt} + y_1 + 2y_{0,tT} + (y_0^2 - 1)y_{0,t} &= 0 \\
y_0(t, T) &= A(T)e^{it} + A^*(T)e^{-it} \\
y_{1,tt} + y_1 + 2\left[iA_T e^{it} - iA_T^* e^{-it}\right] + \left[A^2 e^{2it} + 2|A|^2 + (A^*)^2 e^{-2it} - 1\right]\left[iAe^{it} - iA^* e^{-it}\right] &= 0 \\
y_{1,tt} + y_1 + iA^3 e^{3it} + \left[2iA_T + i|A|^2 A - iA\right] e^{it} + \text{ complex conjugate } &= 0
\end{aligned}$$

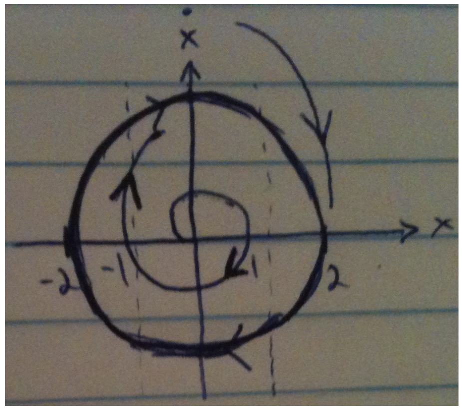
We require that $y_1(t, T)$ is a periodic function of "fast" time t. So we must have

$$\begin{aligned}
A_T + \frac{1}{2}(|A|^2 - a)A &= 0 \\
A(T) &= r(T)e^{i\phi(T)} \\
A_T &= \left[r_T + ir\phi_T\right] e^{i\phi} \\
r_T + ir\phi_T + \frac{1}{2}(r^2 - 1)r &= 0 \\
r_T + \frac{1}{2}r(r^2 - 1) &= 0 \\
\phi_T &= 0 \\
\phi &= \phi_0
\end{aligned}$$



$$\begin{aligned}
x(t;\epsilon) &= A(\epsilon t)e^{it} + \text{ complex conjugate } + O(\epsilon) \\
&= r(\epsilon t)e^{i(t + \phi_0)} + \text{ complex conjugate } + O(\epsilon) \qquad \text{for times } t = O\left(\frac{1}{\epsilon}\right)
\end{aligned}$$

$$\begin{aligned}
r = 0 \quad &\Rightarrow \quad x = 0 \qquad \text{(equilibrium)} \\
r = 1 \quad &\Rightarrow \quad x = 2\cos(t + \phi_0)
\end{aligned}$$



Let's try to formulate an energy argument for this system. Energy equation:

$$\begin{aligned} \dot{x}\ddot{x} + \dot{x}x + \epsilon(x^2-1)\dot{x}^2 &= 0 \\ \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\right) &= -\epsilon(x^2-1)\dot{x}^2 \begin{cases} > 0 & |x| < 1 \text{ (negative damping)} \\ < 0 & |x| > 1 \text{ (positive damping)} \end{cases} \end{aligned}$$

For a periodic solution,

$$\oint (x^2-1)\dot{x}^2\,dt = 0$$

For weak damping:

$$\begin{aligned} x(t) &= a\cos t \\ \int_0^{2\pi} (a^2\cos^2 t - 1)\cdot a^2\sin^2 t\,dt &= 0 \\ \frac{a^2}{2\pi}\int_0^{2\pi} \cos^2 t\cdot\sin^2 t\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\sin^2 t\,dt \\ \frac{1}{2\pi}\int_0^{2\pi}\sin^2 t\,dt &= \frac{1}{2} \\ \frac{1}{2\pi}\int_0^{2\pi}(\cos^2 t\sin^2 t)\,dt &= \frac{1}{2\pi}\int_0^{2\pi}(\sin^2 t - \sin^4 t)\,dt \\ &= \frac{1}{2} - \frac{3}{8} \\ &= \frac{1}{8} \\ \frac{a^2}{8} &= \frac{1}{2} \\ a &= 2 \end{aligned}$$

23 5-23-12

23.1 Method of Averaging

$$
\begin{aligned}
x_t &= \epsilon f(x,t) \\
x(0) &= c \\
x &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
f(x, t+2\pi) &= f(x,t)
\end{aligned}
$$

$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, f is periodic in time.

We introduce multiple scale variables $t,\ T = \epsilon t$. Then

$$
\begin{aligned}
x(t;\epsilon) &= y(t,T;\epsilon)|_{T=\epsilon t} \\
\frac{d}{dt} &\to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T} \\
y_t + \epsilon y_T &= \epsilon f(y,t)
\end{aligned}
$$

We look for solutions that are periodic in t (i.e. no secular terms):

$$
\begin{aligned}
y(t+2\pi, T;\epsilon) &= y(t,T;\epsilon) \\
y &= y_0(t,T) + \epsilon y_1(t,T) + O(\epsilon^2) \\
y_{0,t} + \epsilon y_{1,t} + \epsilon y_{0,T} &= \epsilon f(y_0,t) + O(\epsilon^2) \\
O(1): \qquad y_{0,t} &= 0 \\
y_0 &= y_0(T) \\
O(\epsilon): \qquad y_{1,t} + y_{0,T} &= f(y_0,t) \\
y_1(t+2\pi,T) &= y_1(t,T) \\
0 = \int_0^{2\pi} y_t \, dt &= \int_0^{2\pi} g(t)\, dt \\
\text{Need:} \qquad \overline{g} &= \frac{1}{2\pi}\int_0^{2\pi} g(t)\, dt = 0
\end{aligned}
$$

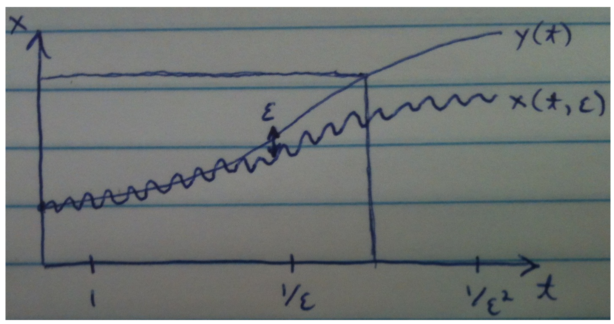
We have

$$
y_{1,t} = -y_{0,T} + f(y_0,t)
$$

The solvability condition is that

$$\frac{1}{2\pi}\int_0^{2\pi}(-y_{0,T}+f(y_0,t))\,dt=0$$

$$\begin{aligned}
y_{0,T} &= \overline{f}(y_0)\\
\overline{f}(y_0) &= \frac{1}{2\pi}\int_0^{2\pi} f(y_0,t)\,dt\\
y(t) &= y_0(\epsilon t)\\
\partial_T &= \frac{1}{\epsilon}\partial_t\\
y_t &= \epsilon\overline{f}(y)\\
\overline{f}(y) &= \frac{1}{2\pi}\int_0^{2\pi} f(y,t)\,dt\\
x_t &= \epsilon f(x,t)
\end{aligned}$$



Theorem 23.1.

For smooth t-periodic vector fields $f(x,t)$ there exist constants $\epsilon_0, c, k > 0$ such that for all ϵ with $|\epsilon| < \epsilon_0$ we have

$$|x(t;\epsilon) - y(t)| < k\epsilon$$

for $|t| < \frac{c}{\epsilon}$.

23.2 Geometrical Interpretation

$$\begin{aligned}
x_t &= \epsilon f(x,t)\\
p^\epsilon &: \mathbb{R}^n \to \mathbb{R}^n \qquad \text{(Poincaré map)}\\
x(0) &\mapsto x(2\pi)\\
p^\epsilon(x_0) - x_0 &= O(\epsilon)
\end{aligned}$$

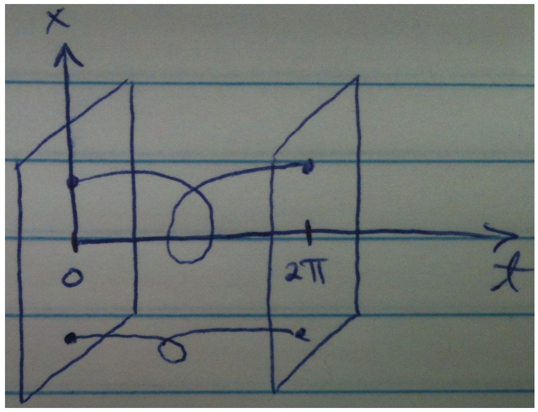


Figure 10: Poincaré map.

The flow of the averaged equation approximates the Poincaré map of the full equation (on times $t = O\left(\frac{1}{\epsilon}\right)$). Hyperbolic fixed points of the averaged equation correspond to 2π-periodic solutions of the full equation (for ϵ sufficiently small) with the same stability.

23.3 Periodic Standard Form

$$
\begin{aligned}
\ddot{y} + y &= \epsilon g(y, \dot{y}, t) \qquad (2\pi\text{-periodic}) \\
y(t) &= x_1(t)\cos t + x_2(t)\sin t \\
\dot{y}(t) &= -x_1(t)\sin t + x_2(t)\cos t \qquad (23.1) \\
\begin{pmatrix} y \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
\ddot{y} &= -x_1\cos t - x_2 \sin t - \dot{x}_1 \sin t + \dot{x}_2 \cos t \\
&= -y - \dot{x}_1 \sin t + \dot{x}_2 \cos t \\
-\dot{x}_1 \sin t + \dot{x}_2 \cos t &= \epsilon g(x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t, t) = \epsilon f(x, t) \\
\dot{x}_1 \cos t + \dot{x}_2 \sin t &= 0 \qquad (\text{so 23.1 holds}) \\
\dot{x}_1 &= -\epsilon (\sin t) f(x, t) \\
\dot{x}_2 &= \epsilon (\cos t) f(x, t) \\
\dot{x} &= \epsilon f(x, t)
\end{aligned}
$$

24 5-25-12

24.1 WKB Method

Simple harmonic oscillator with slowly varying frequency:

$$x_{tt} + \omega^2(\epsilon t)x = 0$$

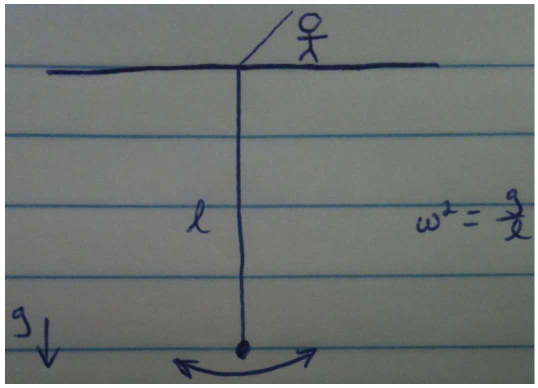


Figure 11: A pendulum system where the length of the pendulum can change.

$$\begin{aligned} T &= \epsilon t \\ \frac{d}{dt} &= \epsilon \frac{d}{dT} \\ \epsilon^2 x_{TT} + \omega^2(T)X &= 0 \end{aligned}$$

Slow vs. small variations in frequency. Here, we use the fact that the variations are slow.

We want to find an approximate solution that is valid for $t = O\left(\frac{1}{\epsilon}\right)$. Try a multiple scale expansion: $t,\ T = \epsilon t$.

$$\begin{aligned} x(t;\epsilon) &= y(t,T;\epsilon)|_{T=\epsilon t} \\ \frac{d}{dt} &\to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T} \\ \frac{d^2}{dt^2} &\to \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial T} + \epsilon^2 \frac{\partial^2}{\partial T^2} \\ y_{tt} + 2\epsilon y_{tT} + \epsilon^2 y_{TT} + \omega^2(T)y &= 0 \\ y &= y_0(t,T) + \epsilon y_1(t,T) + \cdots \\ y_{0,tt} + \omega^2(T)y_0 &= 0 \\ y_{1,tt} + \omega^2(T)y_1 + 2y_{0,tT} &= 0 \\ y_0(t,T) &= A(T)e^{i\omega(T)t} + A^*(T)e^{-i\omega(T)t} \\ y_{0,t} &= i\omega A e^{i\omega t} + \text{ complex conjugate} \\ y_{0,tT} &= i(\omega A)_T e^{i\omega t} - \omega\omega_T A t e^{i\omega t} + \text{ complex conjugate} \\ y_{1,tt} + \omega^2 y_1 &= 2\omega\omega_T A t e^{i\omega t} - i(\omega A)_T e^{i\omega t} + \text{ complex conjugate} \end{aligned}$$

We get secular terms, and the solutions is not valid for long times $t = O\left(\frac{1}{\epsilon}\right)$.

Problem: the period is changing on a slow time-scale.

We've got oscillations with phase $\omega(T)t = \omega(\epsilon t)t$. The right way to do this is to use a "fast" phase

$$\begin{aligned}\theta &= \frac{\phi(\epsilon t)}{\epsilon}\\ \phi_T(T) &= \omega(T).\end{aligned}$$

WKB expansion:

$$\begin{aligned}x(t;\epsilon) &= y(\theta, T; \epsilon)|_{\theta=\frac{\phi(\epsilon t)}{\epsilon},\ T=\epsilon t}\\ x(t;\epsilon) &= y\left(\frac{\phi(\epsilon t)}{\epsilon}, \epsilon t; \epsilon\right)\\ \frac{dx}{dt} &= \phi_T\frac{\partial y}{\partial \theta} + \epsilon\frac{\partial y}{\partial T}\\ \frac{d^2x}{dt^2} &= \phi_T\left[\phi_T\frac{\partial^2 y}{\partial\theta^2} + \epsilon\frac{\partial^2 y}{\partial T\partial\theta}\right] + \epsilon\phi_{TT}\frac{\partial y}{\partial\theta} + \epsilon\left[\phi_T\frac{\partial^2 y}{\partial\theta\partial T} + \epsilon\frac{\partial^2 y}{\partial T^2}\right]\\ &= \phi_T^2 y_{\theta\theta} + \epsilon\left[\phi_{TT}y_\theta + 2\phi_T y_{\theta T}\right] + \epsilon^2 y_{TT}\end{aligned}$$

$$\phi_T^2 y_{\theta\theta} + \epsilon\left[\phi_{TT}y_\theta + 2\phi_T y_{\theta T}\right] + \epsilon^2 y_{TT} + \omega^2(T)y = 0$$

Expand:

$$y = y_0(\theta, T) + \epsilon y_1(\theta, T) + \cdots$$

Require: $y(\theta, T; \epsilon)$ is a 2π-periodic function of θ.

$$\begin{aligned}\phi_T^2 y_{0,\theta\theta} + \omega^2 y_0 &= 0\\ \phi_T^2 y_{1,\theta\theta} + \omega^2 y_1 + \phi_{TT}y_{0,\theta} + 2\phi_T y_{0,\theta T} &= 0\\ &\vdots\end{aligned}$$

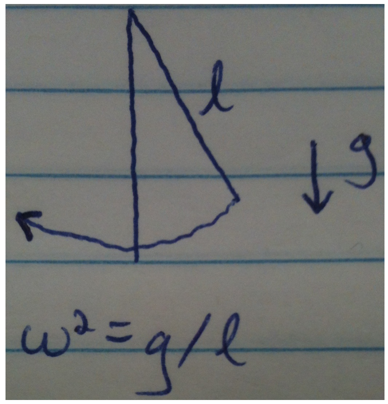
$y_0(\theta, T)$ is 2π-periodic in θ if and only if $\phi_T^2 = \omega^2$, or $\phi_T = \pm\omega$.

$$\begin{aligned}y_0 &= A(T)e^{i\phi} + A^*(T)e^{-i\theta}\\ \omega^2(y_{1,\theta\theta} + y_1) + \phi_{TT}(iAe^{i\phi} + \text{ c.c. }) + 2\phi_T(iA_Te^{i\phi} + \text{ c.c. }) &= 0\\ \omega^2(y_{1,\theta\theta} + y_1) + \underbrace{i(2\phi_T A_T + \phi_{TT}A)}_{=0 \text{so } y \text{ is } 2\pi\text{-periodic}}e^{i\phi} + \text{ c.c. } &= 0\\ 2\phi_T A_T + \phi_{TT}A &= 0\\ \phi_T &= \omega\\ (\omega|A|^2)_T &= 0\end{aligned}$$

25 5-30-12

25.1 WKB Method

$$\ddot{x} + \omega^2(\epsilon t)x = 0$$



$$\begin{aligned}
x(t;\epsilon) &= A(\epsilon t)e^{i\phi(\epsilon t)/\epsilon} \\
T &= \epsilon t \\
\theta &= \frac{\phi(\epsilon t)}{\epsilon} \\
\dot{x} &= (i\phi' A + \epsilon A')e^{i\phi/\epsilon} \\
&\text{primes denote } \frac{d}{dT} \\
\ddot{x} &= i\phi'(i\phi' A + \epsilon A')e^{i\phi/\epsilon} + (\epsilon i\phi'' A + \epsilon i\phi' A' + \epsilon^2 A'')e^{i\phi/\epsilon} \\
&= \left[-(\phi')^2 A + i\epsilon(2\phi' + \phi'' A) + \epsilon^2 A''\right] e^{i\phi/\epsilon} \\
0 &= -(\phi')^2 A + i\epsilon(2\phi' A' + \phi'' A) + \epsilon^2 A'' + \omega^2 A
\end{aligned}$$

Choose $(\phi')^2 = \omega^2$ to eliminate leading-order terms.

$$2\phi' A' + \phi'' A = i\epsilon A''$$

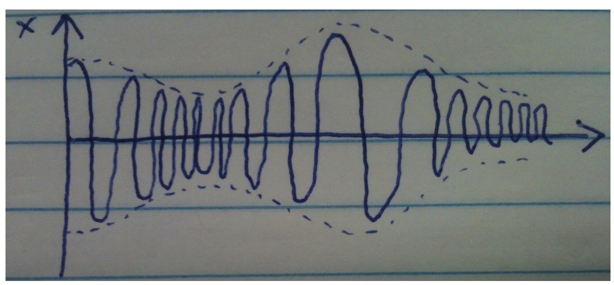
(Liouville-Green)

So far we haven't made any approximations. Let's look for an expansion

$$\begin{aligned}
A &= A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots \\
2\phi' A_0' + \phi'' A_0 &= 0
\end{aligned}$$

Let's say we choose $\phi' = \omega$.

$$
\begin{aligned}
A_0(T) &= \frac{1}{2}a(T)e^{i\delta} \\
2\omega a' + \omega' a &= 0 \\
\frac{a'}{a} &= -\frac{\omega'}{2\omega} \\
\log a &= -\frac{1}{2}\log(\omega) + c \\
a &= \frac{a_0}{\sqrt{\omega}} \\
x &= A_0(T)e^{i\phi/\epsilon} + \text{complex conjugate} + O(\epsilon) \\
&= \frac{1}{2}ae^{i\delta}e^{i\phi/\epsilon} + \text{complex conjugate} + O(\epsilon) \\
x &= a\cos\left(\frac{\phi}{\epsilon} + \delta\right) + O(\epsilon) \\
\phi(T) &= \int_0^T \omega(\hat{T})\,d\hat{T} \\
\omega a^2 &= \text{constant}
\end{aligned}
$$



$$
\begin{aligned}
x &= a(\epsilon t)\cos\left[\frac{\phi(\epsilon t)}{\epsilon}\right] & t_0 &= O\left(\frac{1}{\epsilon}\right) \\
&= a(\epsilon t_0 + \epsilon s)\cos\left[\frac{\phi(\epsilon t_0 + \epsilon s)}{\epsilon}\right] & s &= O(1) \\
& & t &= t_0 + s \\
&= a(\epsilon t_0)\cos\left[\frac{1}{\epsilon}\left[\phi(\epsilon t_0) + \epsilon\phi'(\epsilon t_0)s + O(\epsilon^2)\right]\right] \\
&\sim a(\epsilon t_0)\cos\left[\frac{\phi(\epsilon t_0)}{\epsilon} + \omega(\epsilon t_0)s\right]
\end{aligned}
$$

ωa^2 is conserved under slow variations in ω. For this reason, we say that ωa^2 is adiabatic invariant, and we call it the action.

$$\begin{aligned}
\text{Energy } &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 = E \\
\dot{x} &= -a\phi' \sin\left(\frac{\phi}{\epsilon} + \delta\right) + O(\epsilon) \\
&= -a\omega \sin\left(\frac{\phi}{\epsilon} + \delta\right) + O(\epsilon) \\
x &= a\cos\left(\frac{\phi}{\epsilon} + \delta\right) + O(\epsilon) \\
\text{Energy } &= \frac{1}{2}a^2\omega^2 + O(\epsilon) \\
\text{Action } &= \frac{1}{2}\omega a^2 = \frac{E}{\omega}
\end{aligned}$$

There's an interesting quantum mechanical interpretation of the action involving energy levels.

25.2 Schrödinger Equation

$$\begin{aligned}
i\hbar\Psi_t &= -\frac{\hbar}{2m}\Psi_{xx} + V(x)\Psi \\
\Psi(x,t) &= \phi(x)e^{-iEt/\hbar} \\
-\frac{\hbar^2}{2m}\phi_{xx} + V(x)\phi &= E\phi \\
\frac{\hbar^2}{2m}\phi_{xx} + [E - V(x)]\phi &= 0
\end{aligned}$$

$\hbar \to 0$ corresponds to the WKB approximation, and this is called the semiclassical limit.

26 6-1-12

26.1 WKB Method and Turning Points

$$\begin{aligned}
\epsilon^2 y'' + q(x)y &= 0 \\
y &\sim a(x)e^{\phi(x)/\epsilon} \\
(\phi')^2 + q &= 0 \\
\phi' &= \pm\sqrt{-q}
\end{aligned}$$

$$\begin{aligned}
q > 0 \quad &\Rightarrow \quad \phi' = \pm i\sqrt{q}, \quad \phi = \pm iS \\
y &\sim ae^{\pm iS(x)/\epsilon}
\end{aligned}$$

$$\begin{aligned}
q < 0 \quad &\Rightarrow \quad \phi' = \pm\sqrt{-q}, \quad \phi = \pm S \\
y &\sim ae^{\pm S(x)/\epsilon}
\end{aligned}$$

A *turning point* is where $q(x) = 0$, $x \in \mathbb{R}$. At a simple zero ($x = 0$ is a turning point):

$$q(x) = cx + O(x^2).$$

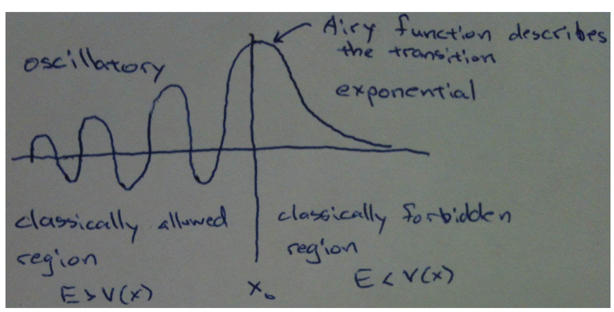
the behavior changes from oscillatory to exponential. Airy equation:

$$y'' + xy = 0$$

The solutions are Airy functions: $Ai(x)$ and $Bi(x)$. Note: the A stands for area, and B follows A.

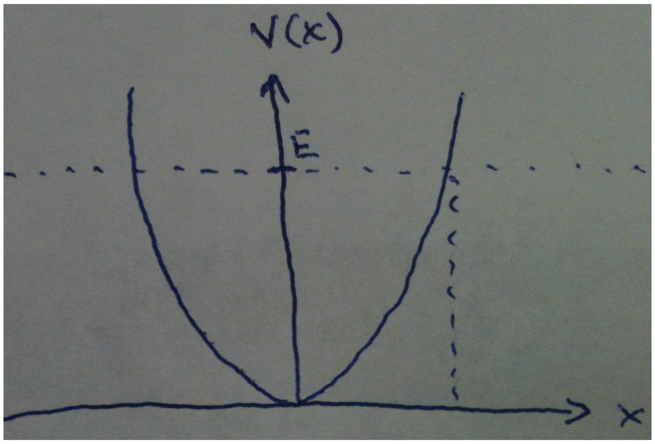
Let's say

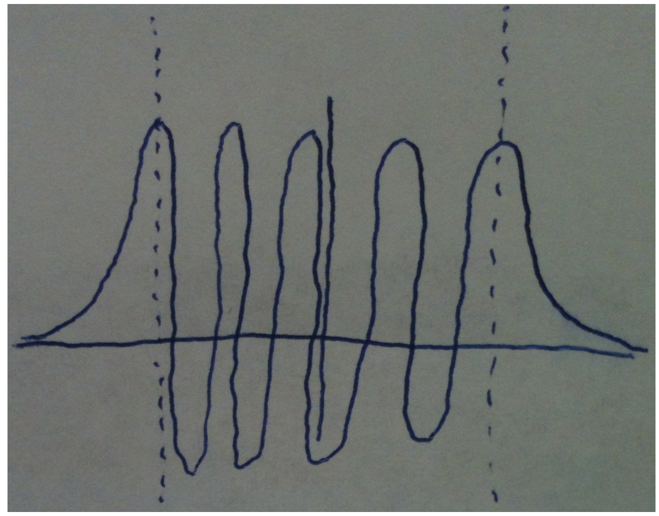
$$\begin{aligned}
q(x) &> 0 \qquad \text{when } x < x_0 \\
q(x) &< 0 \qquad \text{when } x > x_0 \\
\epsilon^2 y'' + q(x)y &= 0
\end{aligned}$$



Schrödinger equation:

$$\begin{aligned}
i\hbar\Psi_t &= -\frac{\hbar^2}{2m}\Psi_{xx} + V(x)\Psi \\
\Psi(x,t) &= \phi(x)e^{-iEt/\hbar} \\
-\frac{1}{2m}\phi'' + V(x)\phi &= E\phi \\
\phi'' + 2m[E - V(x)]\phi &= 0 \\
\phi(x) &= 2m[E - V(x)]
\end{aligned}$$





26.2 A Model Bifurcation Problem for PDEs

$u(x,t)$ satisfies the following:

$$\begin{aligned} u_t &= u_{xx} + \mu \sin u, \qquad 0 < x < 1, \quad t > 0 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= f(x) \end{aligned}$$

This is a heat equation with a nonlinear heat source, $\mu \sin u$. $\mu \geq 0$ is a (dimensionless) parameter that measures the strength of the nonlinear heat sources.

Consider the equilibrium solution $u = 0$. Is it stable?

1. We start by linearizing the PDE around $u = 0$.

$$\begin{aligned} u_t &= u_{xx} + \mu u, \qquad 0 < x < 1 \\ u(0,t) &= u(1,t) = 0 \end{aligned}$$

   Separate variables.

$$\begin{aligned} u(x,t) &= e^{\sigma_n t} \sin(n\pi x), \qquad n = 1, 2, 3, \ldots \\ \sigma_n &= -n^2\pi^2 + \mu \end{aligned}$$

   $\sigma_n < 0$ for all n if $\mu < \pi^2$ ($u = 0$ is linearly stable). $\sigma_1 > 0$ if $\mu > \pi^2$ ($u = 0$ is linearly unstable).

2. How does the nonlinearity affect instability?
   Assume μ is close to π^2. Linear growth rate: $\sigma = \mu - \pi^2$ is small.

$$\underbrace{u_t}_{\epsilon\sigma} = u_{xx} + \mu \left(u - \frac{1}{6} \underbrace{u^3}_{\epsilon^3} + \cdots \right), \qquad u = O(\epsilon)$$

   For a dominant balance between linear growth and nonlinearity, we expect

$$\begin{aligned} \epsilon\sigma &= \epsilon^3 \\ \sigma &= O(\epsilon^2) \end{aligned}$$

This suggests the following expansion:

$$
\begin{aligned}
u &= \epsilon u_1(x,T) + \epsilon^3 u_3(x,T) + O(\epsilon^5) \\
\mu &= \pi^2 + \epsilon^2 \mu_2 + O(\epsilon^4) \\
T &= \epsilon^2 t
\end{aligned}
$$

27 6-4-12

27.1 Model PDE Bifurcation Problem

$$\begin{aligned} u_t &= u_{xx} + \mu \sin u, \qquad 0 < x < 1,\ t > 0 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

- $u(x,t) =$ temperature
- $\mu =$ strength of the source

$u = 0$ is

- linearly stable for $\mu < \pi^2$
- linearly unstable for $\mu > \pi^2$

Look at the effect of nonlinearity near the point of marginal stability, $\mu = \pi^2$. The dominant blance suggested

$$\begin{aligned} \mu - \pi^2 &= O(\epsilon^2) \\ u &= O(\epsilon) \\ \text{time-scales} \quad t &= O\left(\frac{1}{\epsilon^2}\right) \end{aligned}$$

Expand:

$$\begin{aligned} \mu &= \pi^2 + \epsilon^2 \mu_2 + O(\epsilon^4) \\ u &= \epsilon u_1(x,T) + \epsilon^3 u_2(x,T) + O(\epsilon^5) \\ T &= \epsilon^2 t \\ \partial_t &= \epsilon^2 \partial_T \end{aligned}$$

$$\begin{aligned} \epsilon^2 u_T &= u_{xx} + (\pi^2 + \epsilon^2 \mu_2) \sin u, \qquad 0 < x < 1,\ T > 0 \\ u(0,t) &= u(1,t) = 0 \\ \sin u &= u - \frac{1}{6} u^3 + O(u^5) \\ &= \epsilon u_1 + \epsilon^3 u_3 - \frac{1}{6} \epsilon^3 u_1^3 + O(\epsilon^5) \\ \epsilon^3 u_{1,T} + \cdots &= \epsilon u_{1,xx} + \epsilon^3 u_{3,xx} + (\pi^2 + \epsilon^2 u_2) \left(\epsilon u_1 + \epsilon^3 \left[u_3 - \frac{1}{6} u_1^3 \right] + \cdots \right) \\ O(\epsilon): \qquad u_{1,xx} + \pi^2 u_1 &= 0 \\ u_1(0,t) &= u_1(1,t) = 0 \\ O(\epsilon^3): \qquad u_{3,xx} + \pi^2 u_3 &= u_{1,T} + \frac{\pi^2}{6} u_1^3 - \mu_2 u_1 \\ u_3(0,t) &= u_3(1,t) = 0 \end{aligned}$$

We get

$$\begin{aligned}
u_1 &= a(T)\sin(\pi x) \\
u_{3,xx} + \pi^2 u_3 &= a_{1,T}\sin(\pi x) + \frac{\pi^2}{6}a^3\sin^3(\pi x) - \mu_2 a\sin(\pi x) \\
u_3(0,t) = u_3(1,t) &= 0 \\
Lu_3 &= f(x) \\
L &= \frac{d^2}{dx^2} + \pi^2
\end{aligned}$$

This is solvable if for ϕ such that $L\phi = 0$, we have that

$$\begin{aligned}
\langle \phi, Lu_3 \rangle &= \langle \phi, f \rangle \\
\langle L\phi, u_3 \rangle &= \langle \phi, f \rangle \\
0 &= \langle \phi, f \rangle
\end{aligned}$$

Thus, we must have that

$$\begin{aligned}
\langle \sin x, f \rangle &= 0 \\
a_T \underbrace{\left[\int_0^1 \sin^2(\pi x)\,dx\right]}_{=\frac{1}{2}} + \frac{\pi^2}{6}a^3 \underbrace{\left[\int_0^1 \sin^4(\pi x)\,dx\right]}_{=\frac{3}{8}} - \mu_2 a \underbrace{\left[\int_0^1 \sin^2(\pi x)\,dx\right]}_{=\frac{1}{2}} &= 0 \\
\frac{1}{2}a_T + \frac{\pi^2}{16}a^3 - \frac{1}{2}\mu_2 a &= 0 \\
a_T - \mu_2 a + \frac{\pi^2}{8}a^3 &= 0
\end{aligned}$$

This is typically called an *amplitude equation* (Laundau-Stuart). The equilibria are:

$$a = 0 \qquad \text{OR} \qquad a^2 = \frac{8\mu_2}{\pi^2}$$

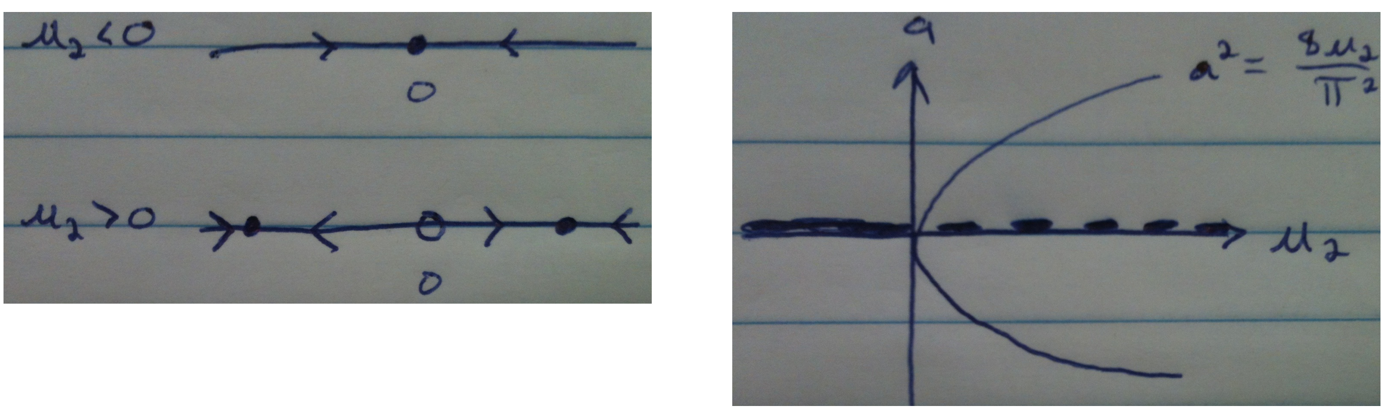


Figure 12: This is a (supercritical) pitchfork bifurcation.

A rigorous analysis of the equilibrium states is obtained using Liapunov-Schmidt reduction.

Initial layer: take

$$
\begin{aligned}
t &= O(1) \\
\mu &= \pi^2 + \epsilon^2 \mu_2 \\
u &= \epsilon u_1(x,t) + \epsilon^3 u_3(x,t) + \cdots \\
u_{1,t} &= u_{1,xx} + \pi^2 u_1 \\
u_1(0,t) &= u_1(1,t) = 0 \\
u_1(x,0) &= f(x) \\
u_1(x,t) &= \sum_{n=1}^{\infty} c_n e^{-(n^2-1)\pi^2 t} \sin(n\pi x) \\
&= c_1 \sin(\pi x) + \sum_{n=2}^{\infty} c_n e^{-(n^2-1)\pi^2 t} \sin(n\pi x) \\
c_n &= 2 \int_0^1 f(x) \sin(n\pi x)\, dx
\end{aligned}
$$

As $t \to \infty$,

$$u_1 \sim c_1 \sin(\pi x).$$

So we require

$$
\begin{aligned}
a(T) &\to c_1 \qquad \text{as } T \to 0 \\
a(0) &= 2 \int_0^1 f(x) \sin(\pi x)\, dx
\end{aligned}
$$

28 6-6-12

Final: Tuesday June 12 from 1:30-3:30
Office Hours: Monday 2:30-4:00

28.1 Outline of Topics

1. Dimensional analysis and scaling
   - Buckingham-Pi Theorem
   - Self-similarity
2. Asymptotic expansions
   - o, O notation
   - Asymptotic vs. convergent series
   - Expansion of integrals
   - (Did NOT cover the method of stationary phase or steepest descent)
3. Regular vs. singular perturbation problems
   - Algebraic equations (e.g. polynomials)
   - Dominant balance (distinguished limits)
4. Method of matched asymptotics
   - Construct inner & outer solutions and match them
   - Uniform solutions
   - Initial layer problems (e.g. enzyme dynamics)
   - Slow-fast dynamics in systems of ODE's
   - Boundary layer problems
5. Method of multiple scales
   - Poincaré-Lindstedt method (periodic solutions)
   - Multiple scales (t, T) and applications to oscillations
   - Method of averaging
   - WKB method
   - Fredholm alternative & solvability conditions $\Rightarrow$ these were a unifying theme

The final will probably be 5 questions (roughly one from each topic).

1. Multiple scales
2. Boundary layers
3. Nondimensionalization
4. Asymptotics

For example:

- Nondimensionalize this equation

- Here's a polynomial involving ϵ, find the roots

Most of this is discussed in chapters 1 and 2 of *Applied Mathematics*.

Things to know:

- Taylor expansion for tan

28.2 Sample Problems

Example 28.1. ***Logan 2.1.4***

$$f(y,\epsilon) = \frac{1}{(1+\epsilon y)^{3/2}}$$
$$y = y_0 + \epsilon y_1 + O(\epsilon^2)$$

Expand $f(y,\epsilon)$ in ϵ up to $O(\epsilon^2)$.

$$\begin{aligned}
f(y,\epsilon) &= (1+\epsilon y)^{-3/2} \\
&= 1 - \frac{3}{2}\epsilon y + \frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(\epsilon y)^2 + O(\epsilon^3) \\
&= 1 - \frac{3}{2}\epsilon y + \frac{15}{8}\epsilon^2 y^2 + O(\epsilon^3) \\
&= 1 - \frac{3}{2}\epsilon y_0 + \epsilon^2\left[\frac{15}{8}y_0^2 - \frac{3}{2}y_1\right] + O(\epsilon^3)
\end{aligned}$$

Example 28.2. ***Logan 2.1.5h***

How does $\exp(\tan\epsilon)$ behave as $\epsilon \to 0$? We are supposed to show that $\exp(\tan\epsilon) = O(1)$.

$$\begin{aligned}
f(\epsilon) = O(g(\epsilon)) \quad &\Rightarrow \quad |f(\epsilon)| \le C|g(\epsilon)| \quad \text{for } |\epsilon| < \delta \\
f(\epsilon) = o(g(\epsilon)) \quad &\Rightarrow \quad \left|\frac{f(\epsilon)}{g(\epsilon)}\right| \to 0 \quad \text{as } \epsilon \to 0 \quad (\text{if } g(\epsilon) \ne 0) \\
f(\epsilon) \sim g(\epsilon) \quad &\Rightarrow \quad \left|\frac{f(\epsilon)}{g(\epsilon)}\right| \to 1
\end{aligned}$$

$\sim$ and o each imply O

$$\begin{aligned}
&f(\epsilon) = \sin\left(\frac{1}{\epsilon}\right) \\
&g(\epsilon) = 1 \\
&\quad f = o(g) \qquad \text{as } \epsilon \to 0 \quad (c = 1)
\end{aligned}$$

$$
\begin{aligned}
\exp(\tan\epsilon) &\sim 1 \qquad \text{as } \epsilon \to 0 \\
\exp(\tan\epsilon) - 1 &\sim \epsilon \qquad \text{as } \epsilon \to 0
\end{aligned}
$$

$$
\begin{aligned}
\exp(\tan\epsilon) &= \exp(\epsilon + O(\epsilon^3)) \\
&= 1 + (\epsilon + O(\epsilon^3)) + O(\epsilon^2) \\
&= 1 + O(\epsilon) \\
\lim_{\epsilon\to 0} \exp(\tan\epsilon) &= 1 \\
\exists\, \delta > 0 \quad \text{such that} \quad |\exp(\tan\epsilon) - 1| &\le 1 \qquad \text{for } |\epsilon| < \delta \\
|\exp(\tan\epsilon)| \le 2 \cdot 1 \qquad \text{for } |\epsilon| < \delta
\end{aligned}
$$

Example 28.3. ***Logan 1.2.3***

$$m' = ax^2 - bx^3$$

- $m =$ biomass
- $x =$ linear dimension
- ax^2 is the growth term (proportional to the surface area)
- bx^3 is the eating term (proportional to the volume)

Assume $m = \rho x^3$.

$$
\begin{aligned}
3\rho x^2 x' &= ax^2 - bx^3 \\
x(0) &= x_0
\end{aligned}
$$

Nondimensionalize.

The dimensions are

- $M =$ biomass
- $L =$ length
- $T =$ time

The parameters are

- a, $[a] = \frac{M}{TL^2}$
- b, $[b] = \frac{M}{TL^3}$
- ρ, $[\rho] = \frac{M}{L^3}$
- x_0, $[x_0] = L$

The variables are

- t, $[t] = T$
- x, $[x] = L$

We have 3 dimensions and 4 parameters, so we should have 1 dimensionless parameter. Let's leave x_0 alone and use a, b, ρ to nondimensionalize mass, length, and time.

$$\left[\frac{a}{b}\right] = L$$
$$\left[\rho\frac{a^3}{b^3}\right] = M$$
$$\left[\frac{\rho}{b}\right] = T$$

$$x^* = \frac{x}{a/b}$$
$$t^* = \frac{t}{\rho/b}$$
$$\text{(think)} \qquad 3(x^*)^2(x^*)' = (x^*)^2 - (x^*)^3$$
$$x^*(0) = \frac{bx_0}{a}$$

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