The Generalized Haar-Walsh Transform (GHWT) for Data Analysis on Graphs and Networks

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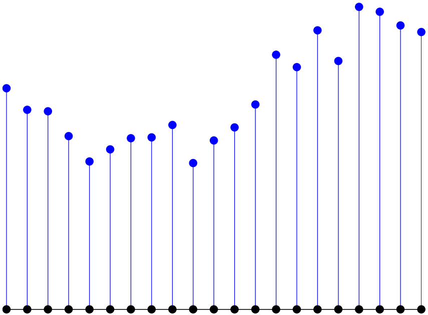
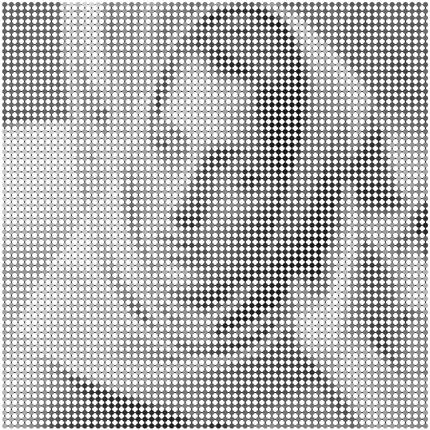
Motivation

Classical signals can be viewed as signals on graphs with simple structures. For example:



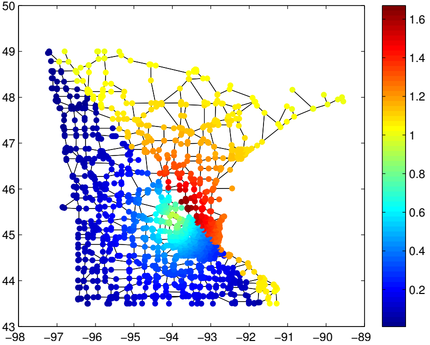
(a) A portion of a 1D signal



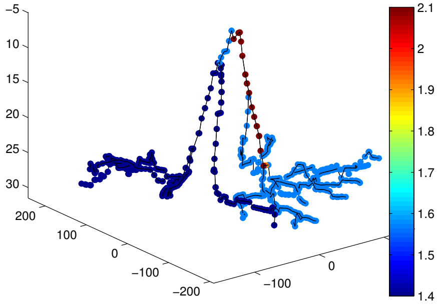
(b) Barbara

Motivation

We wish to extend techniques from classical signal processing to the setting of general graphs. For example:



(a) A mutilated Gaussian signal on the MN road network



(b) Thickness data on a dendritic tree

Motivation

Aims & Objectives:

1. Develop an overcomplete multiscale transform for signals on graphs
2. Develop a corresponding best-basis search algorithm
3. Investigate usefulness for approximation, denoising, classification, function estimation, etc.

Challenges:

1. Irregular structure of the domain
2. Lack of translation, dilation, and a general notion of frequency
   - Critical elements in the wavelet transform
3. Computational complexity!

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3. **Computational complexity!**

Definitions and Notation

Let G be a graph.

- $V = V(G) = \{v_1, \ldots, v_N\}$ is the set of vertices.
- $E = E(G) = \{e_1, \ldots, e_{N'}\}$ is the set of edges, where $e_k = (v_i, v_j)$ represents an edge (or line segment) connecting between adjacent vertices v_i, v_j for some $1 \le i, j \le N$.
- $W = W(G) \in \mathbb{R}^{N \times N}$ is the weight matrix, where w_{ij} denotes the edge weight between vertices i and j.

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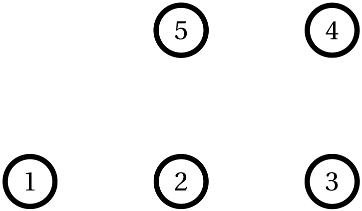

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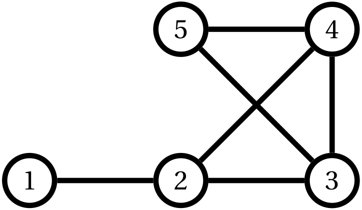

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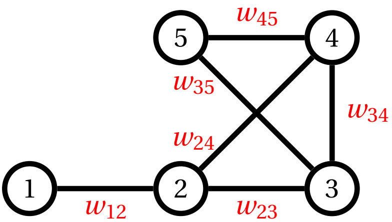

Our Assumptions

In this talk, we assume that the graph is

- **connected.**
- **undirected.** $w_{ij} = w_{ji}$, and thus W is *symmetric*.

Recursive Partitioning

Our transform requires as input a recursive partitioning of the graph.

- We used Fielder vectors of the Laplacian matrices.
- The associated cost is from $O(N \log N)$ to $O(N^2)$, depending on the graph and the implementation.
- More info ⇒ J. Irion, N. Saito: "The Generalized Haar-Walsh Transform," *Proceedings of 2014 IEEE Workshop on Statistical Signal Processing*, pp. 488–491, 2014.

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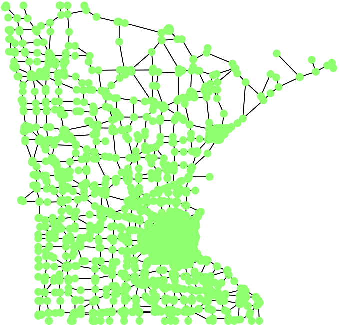

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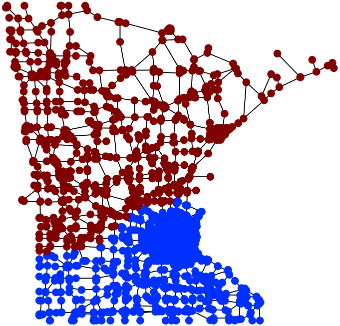

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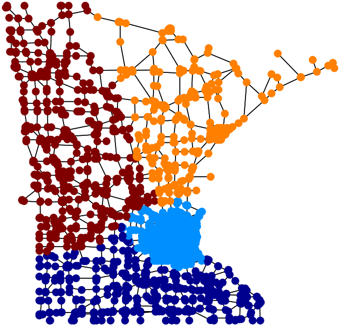

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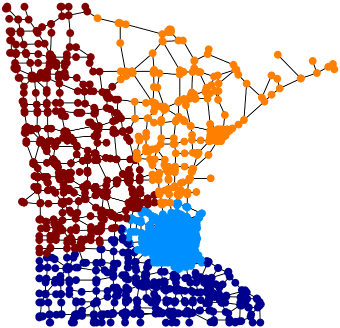

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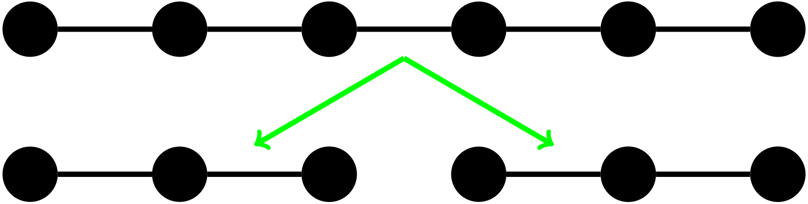

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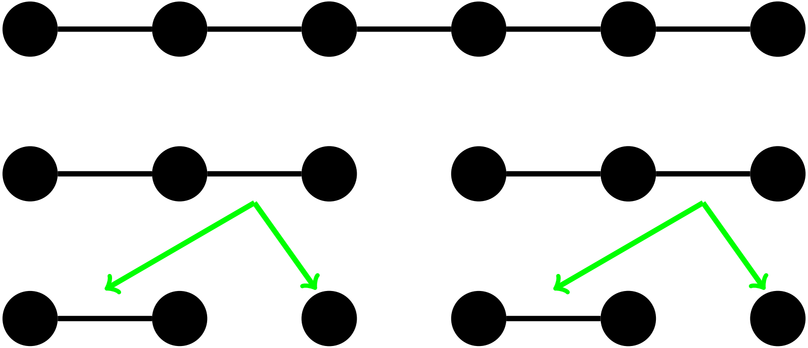

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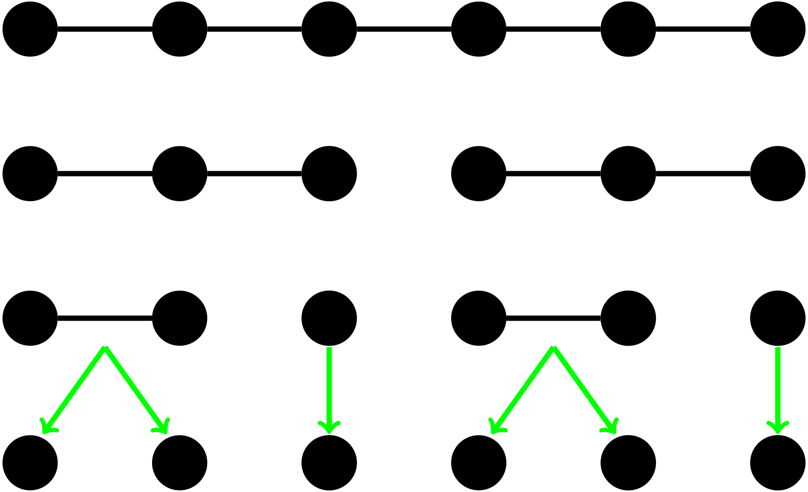

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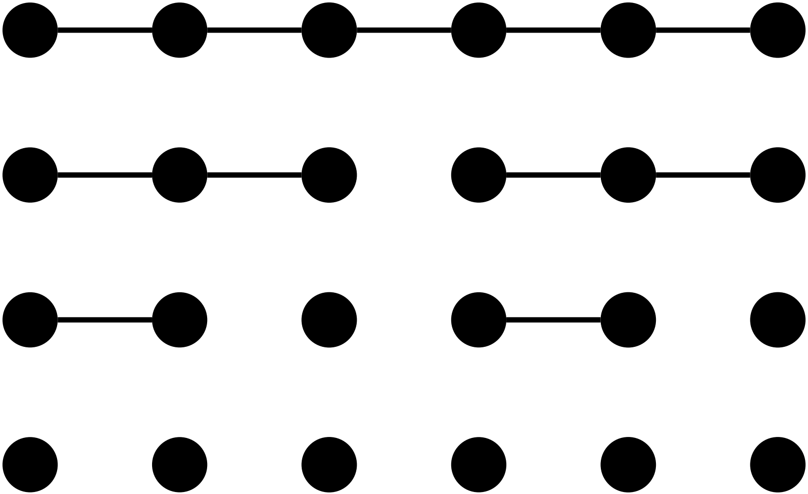

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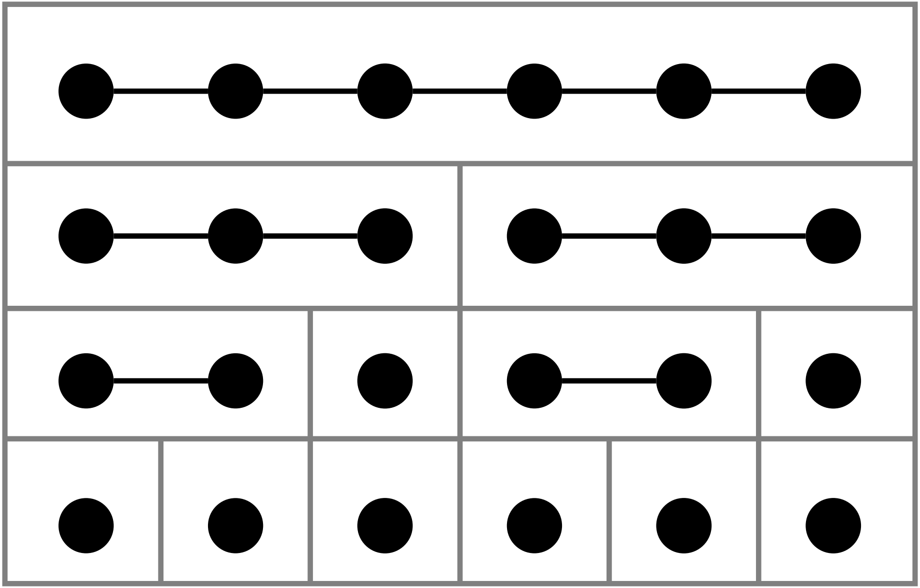

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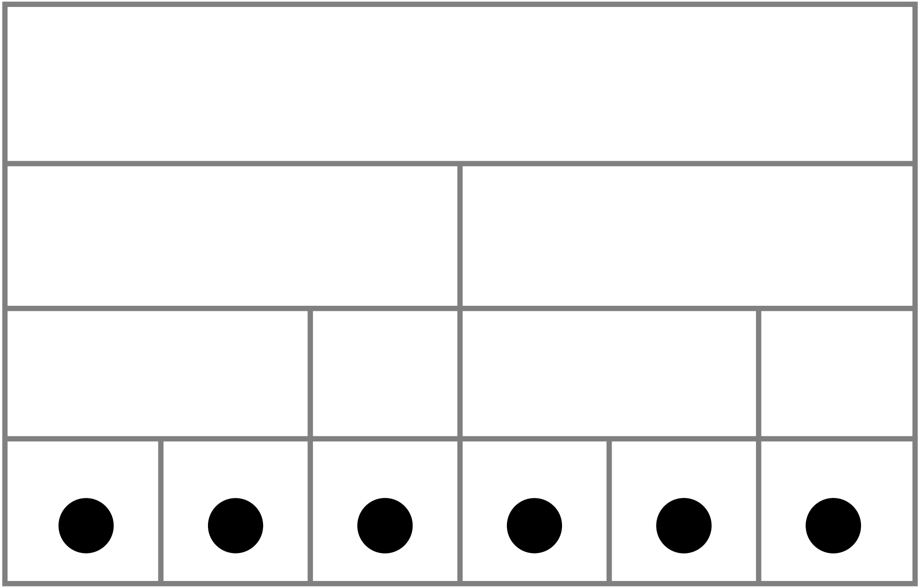

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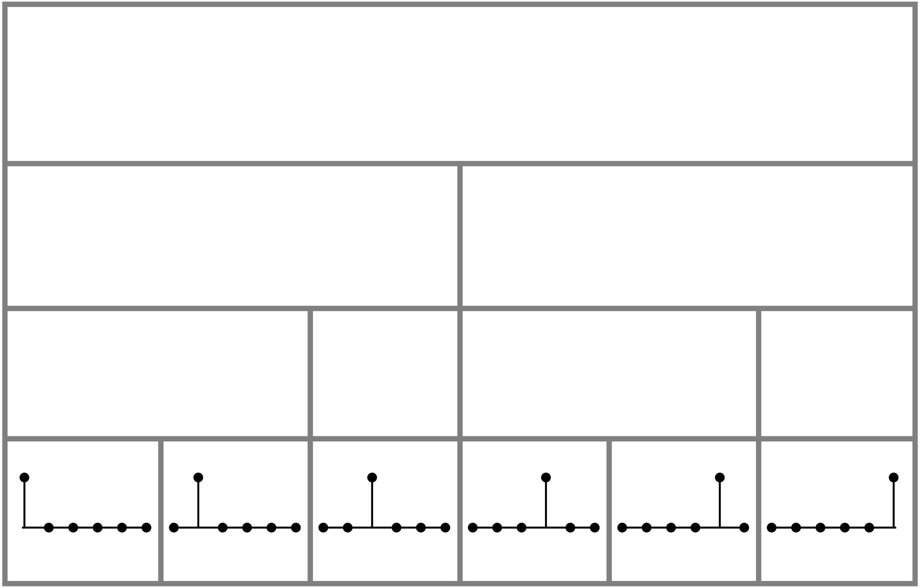

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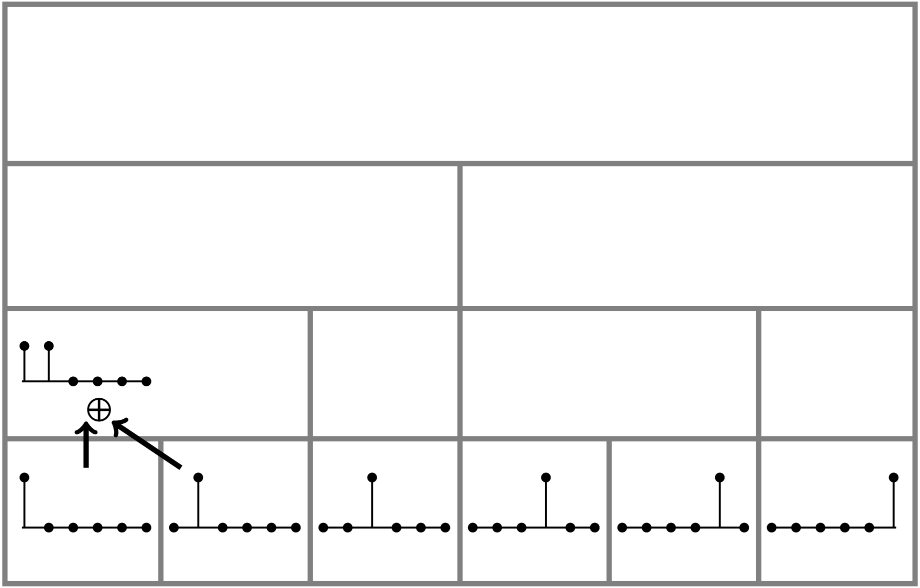

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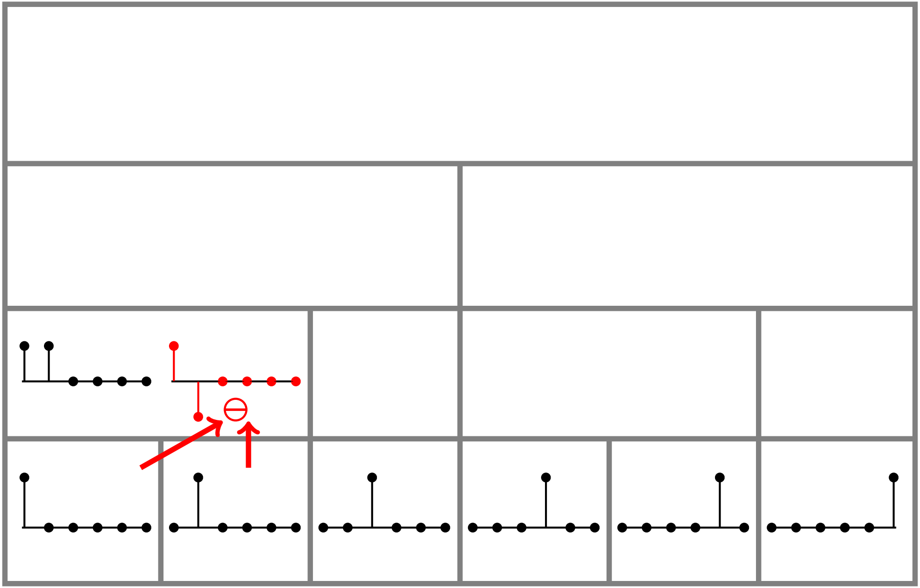

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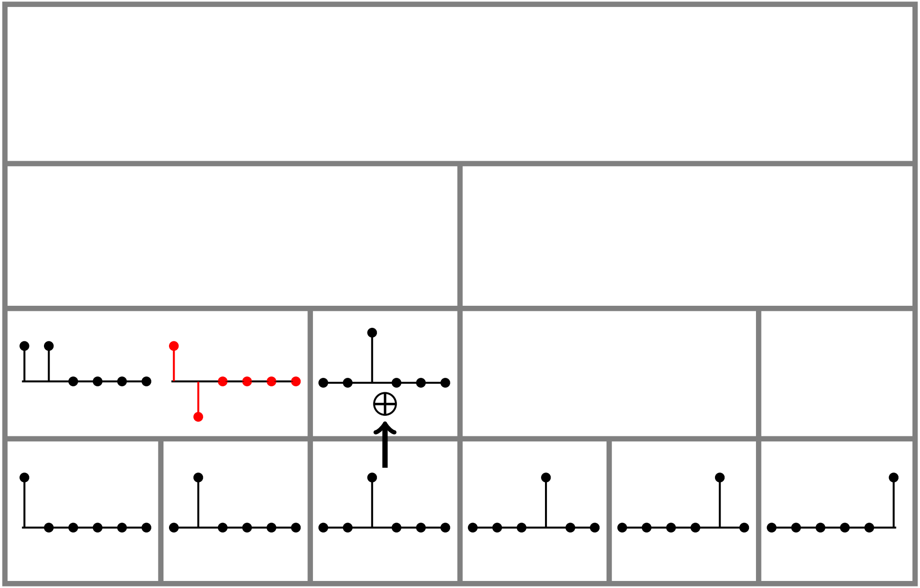

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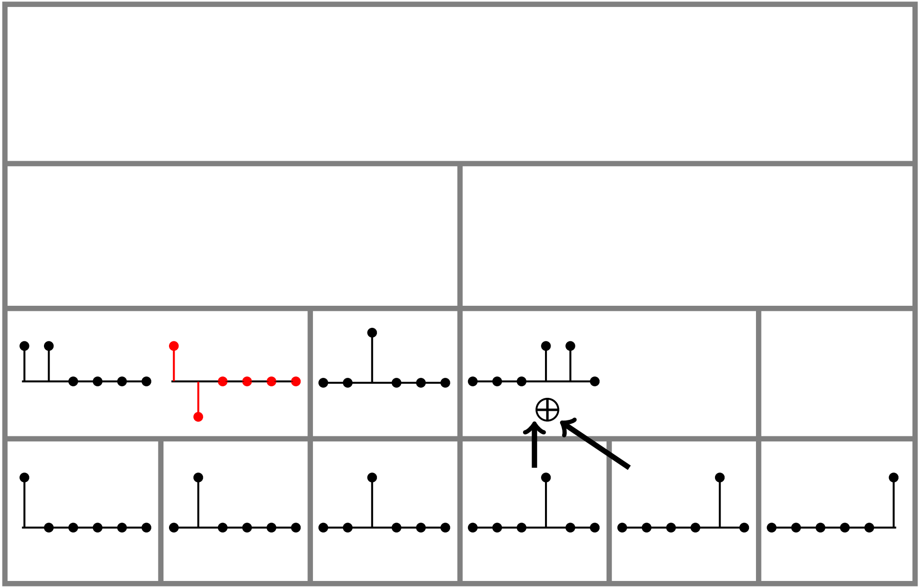

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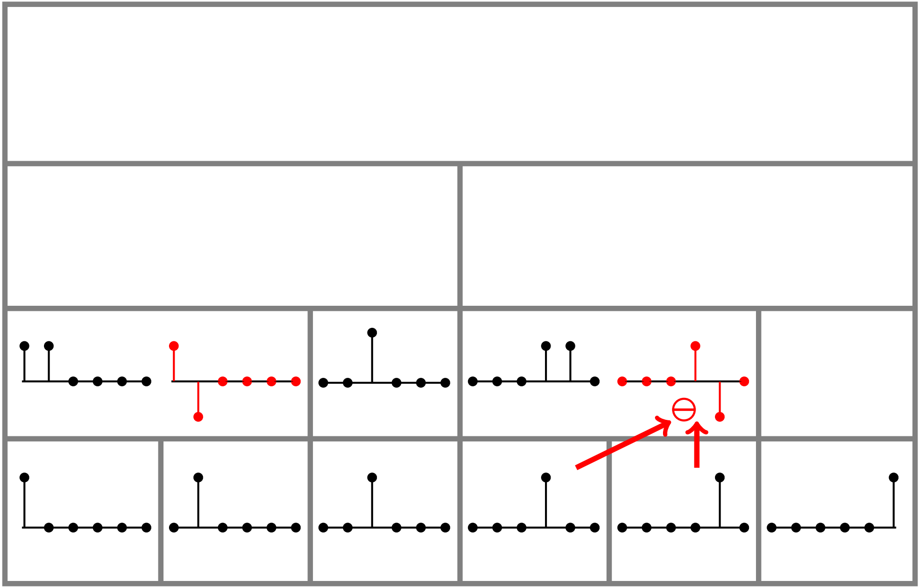

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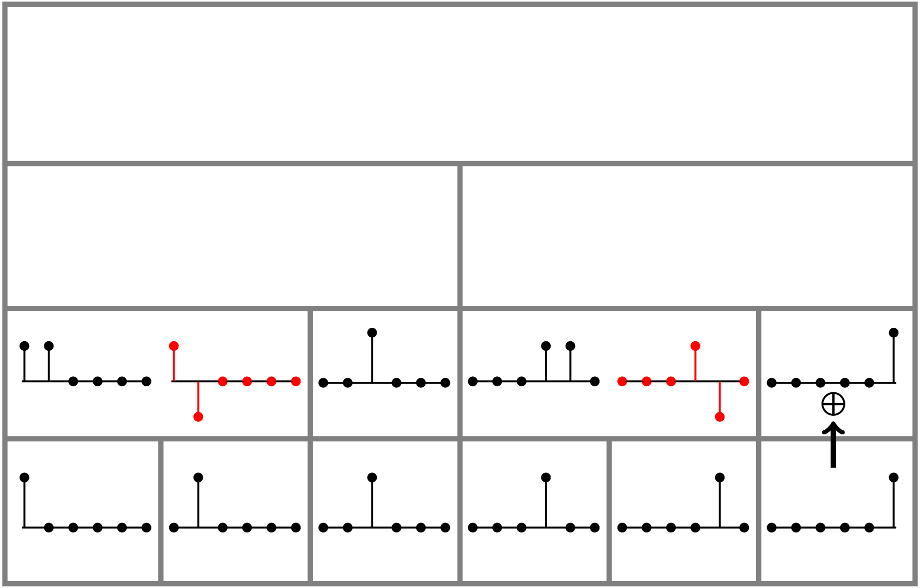

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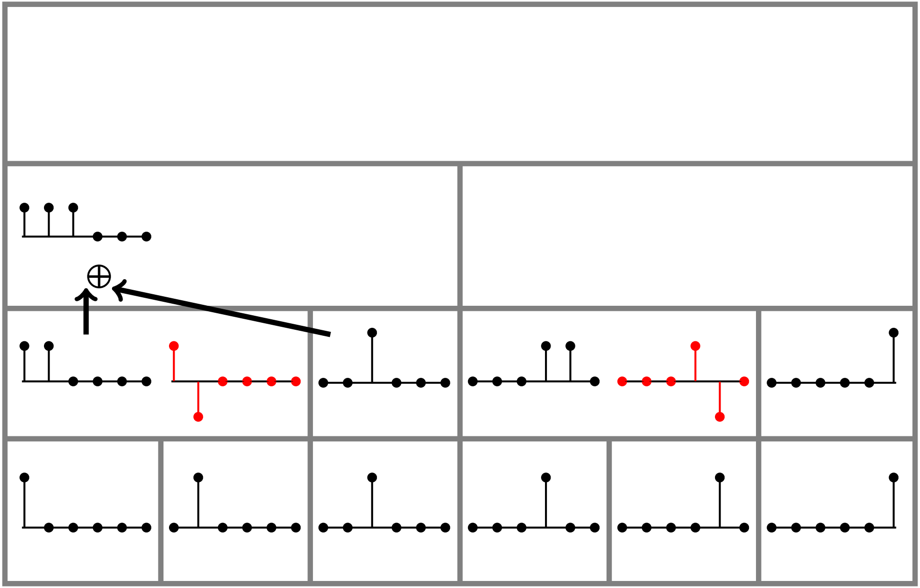

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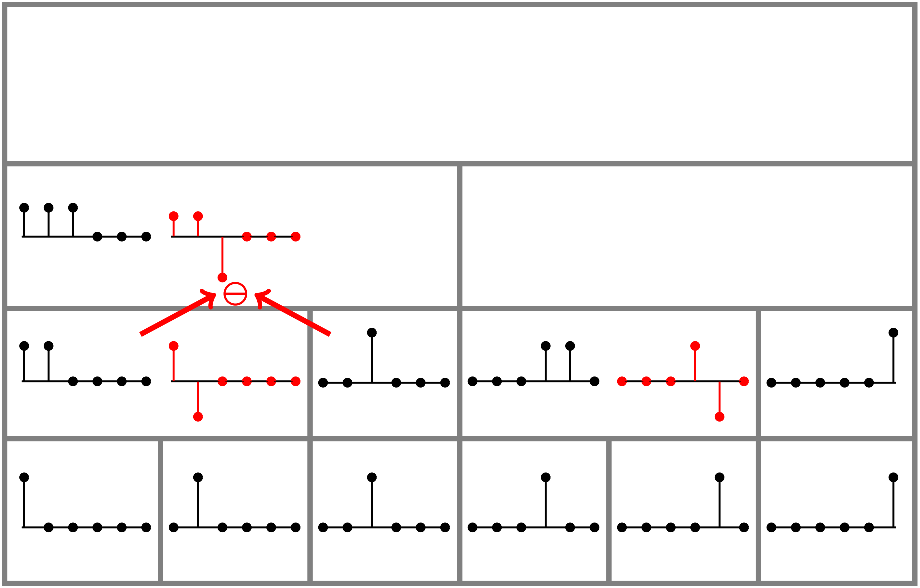

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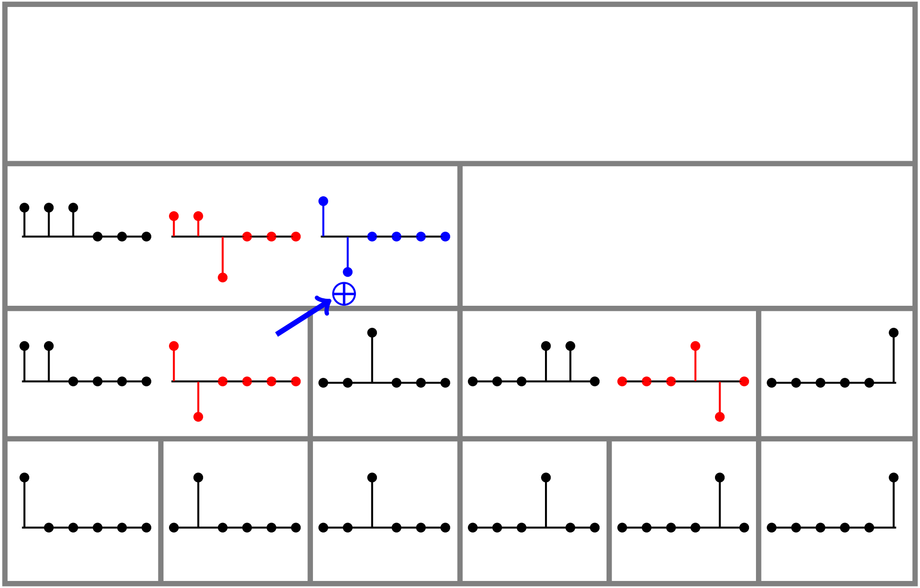

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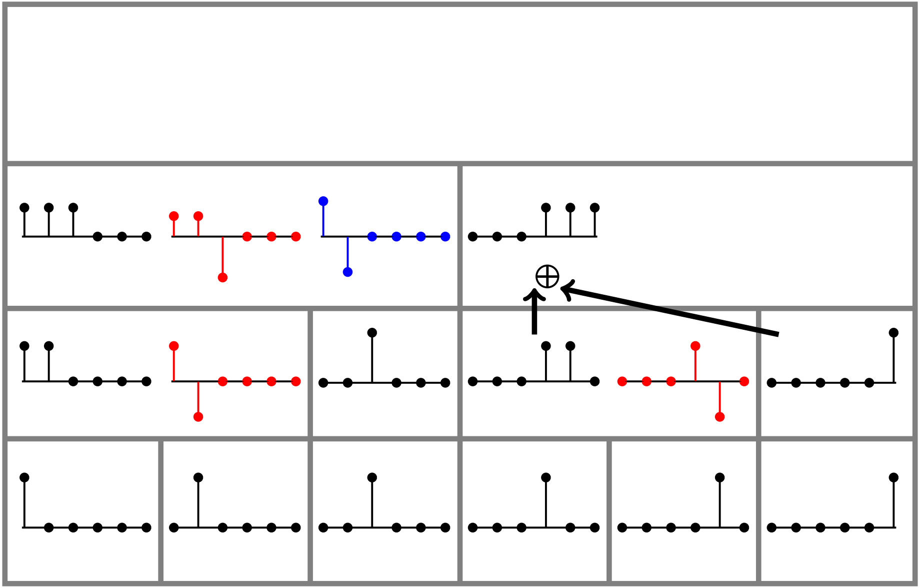

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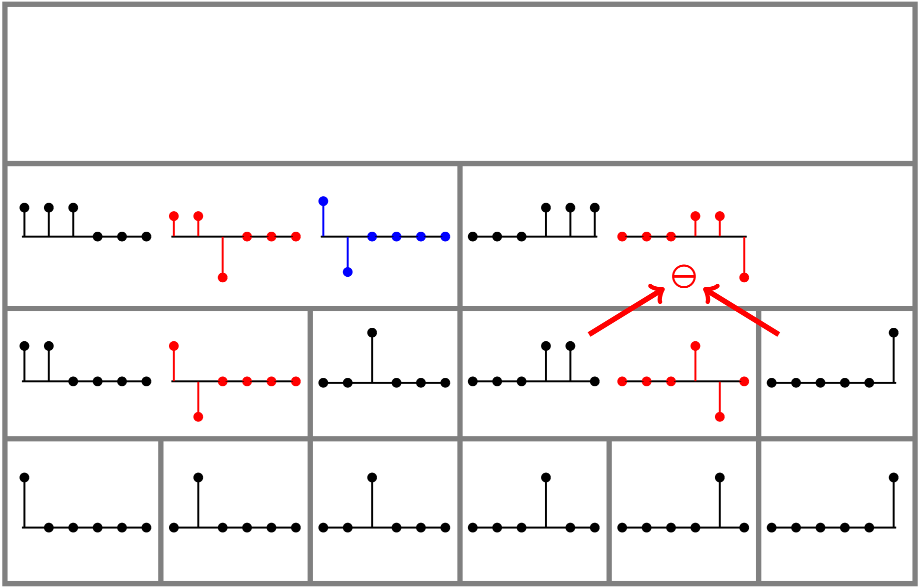

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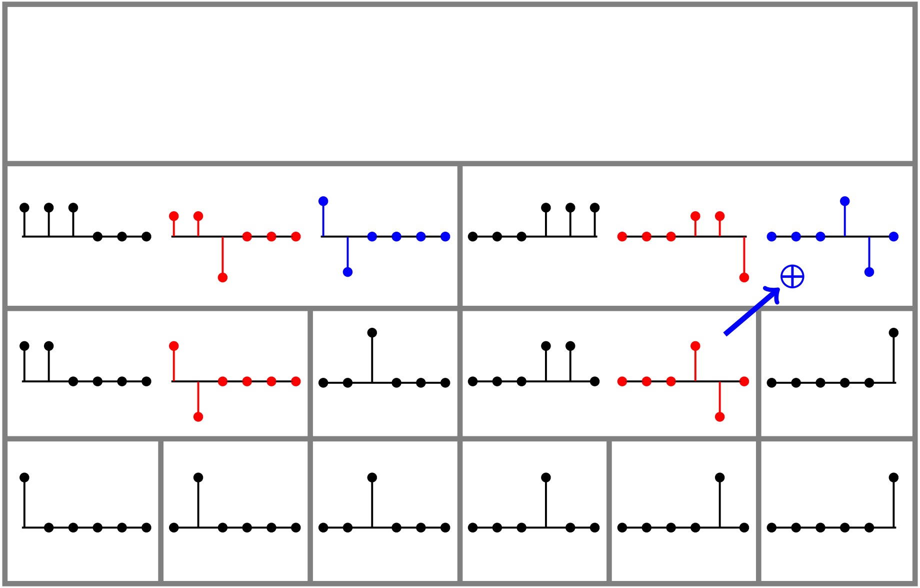

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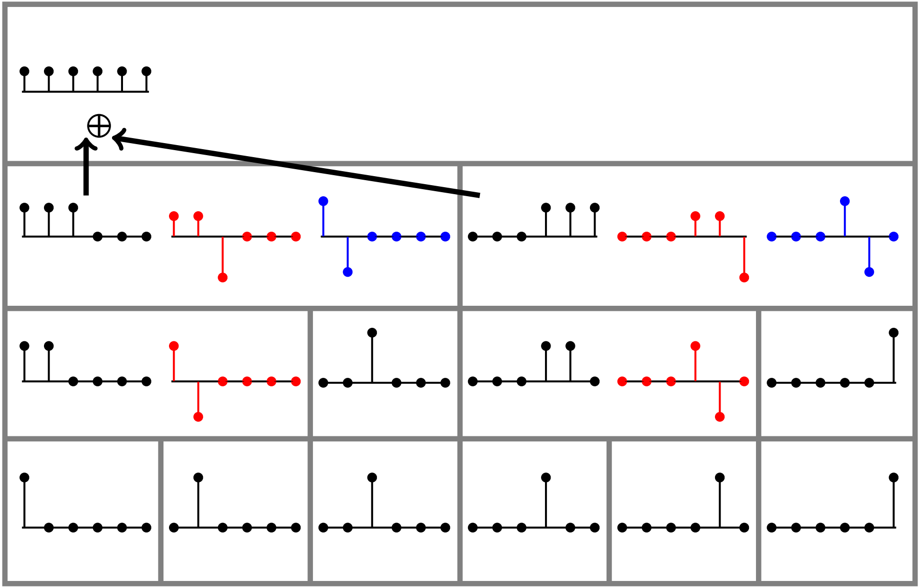

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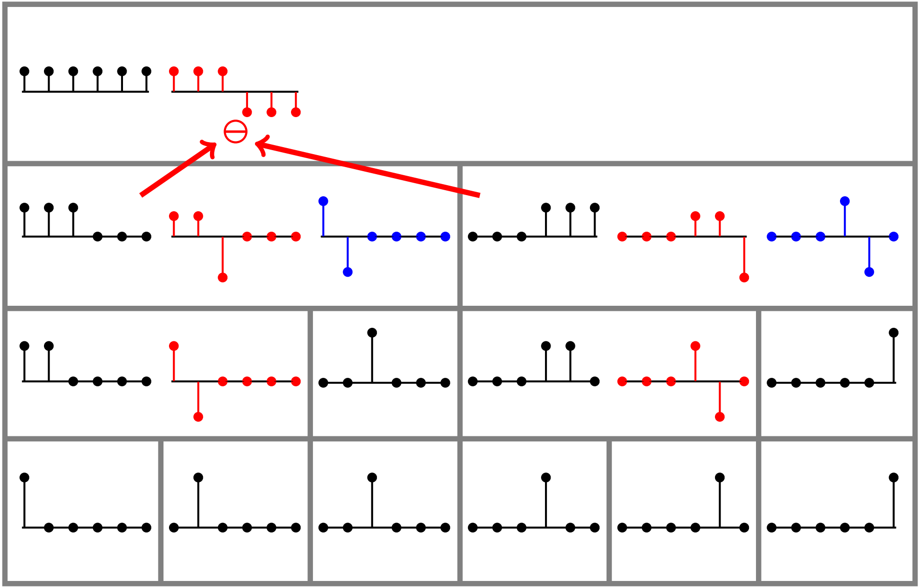

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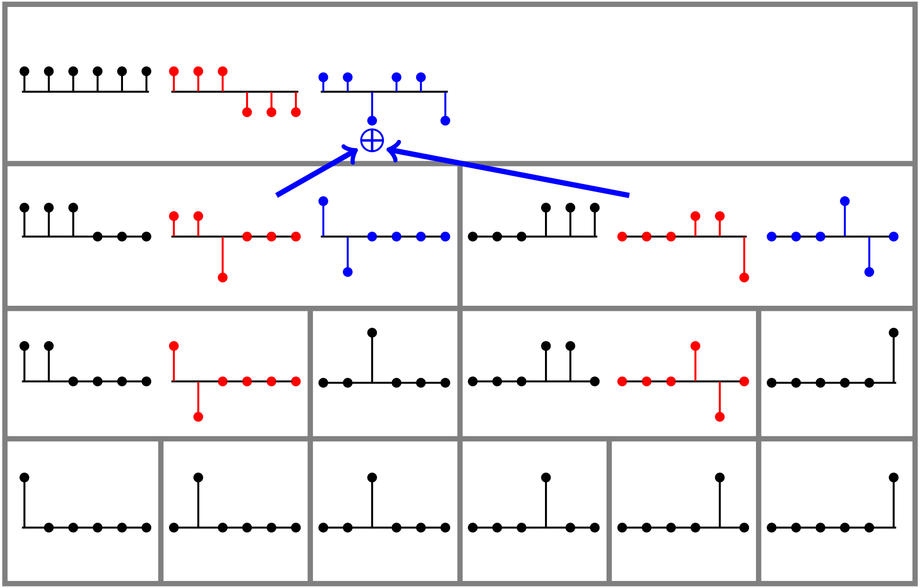

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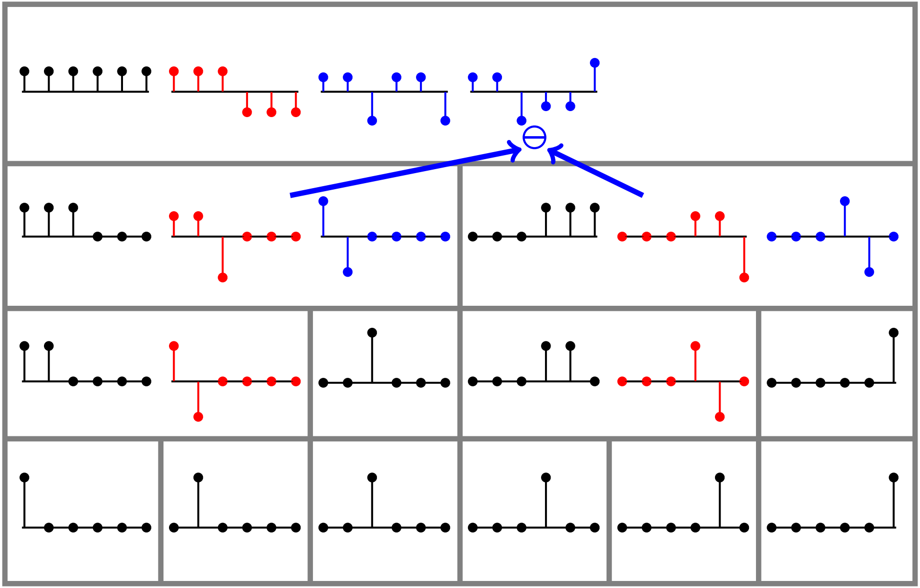

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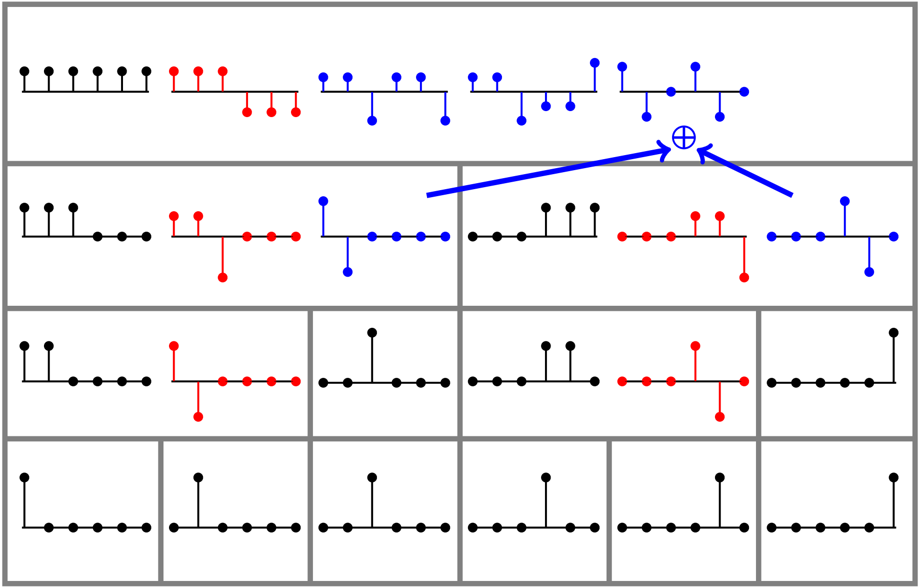

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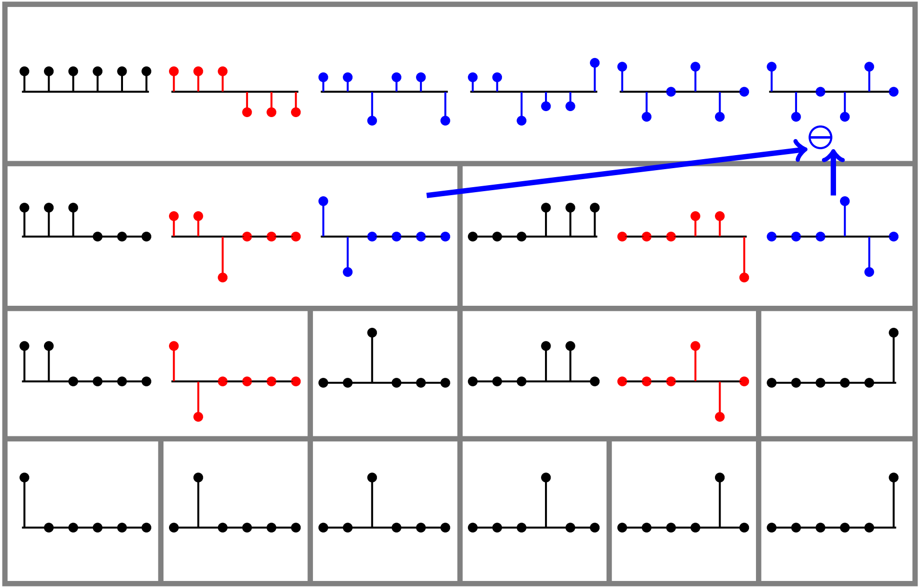

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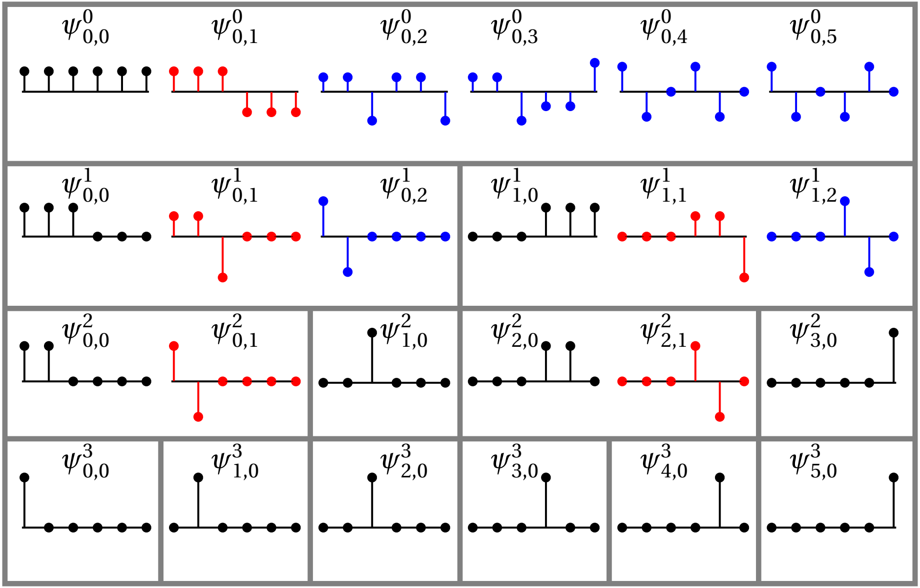
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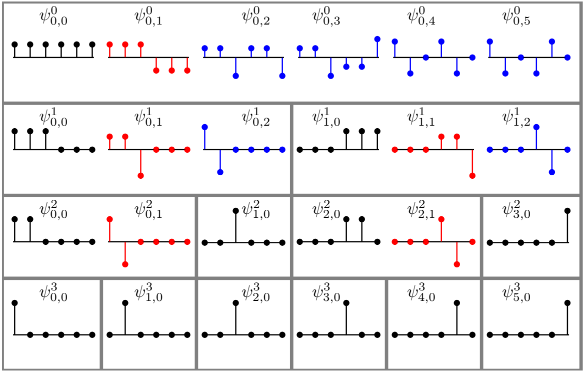
GHWT on P_6 – Coarse-to-Fine Dictionary

We call this the *coarse-to-fine dictionary*.



Notation is $\psi^j_{k,\ell}$, where

- j is the level
- k denotes the region on level j
- ℓ is the *tag*

We have 3 types of basis vectors:

- **scaling vectors** $(\ell = 0)$
- **Haar-like vectors** $(\ell = 1)$
- **Walsh-like vectors** $(\ell \geq 2)$

GHWT Algorithm

Input: a graph partitioning and a signal $f \in \mathbb{R}^N$ on the graph

Output: a dictionary of expansion coefficients $\{d^j_{k,l}\}$

Step 1: Obtain expansion coefficients on level $j = j_{\max}$ ⇔ reorder the original signal

Step 2: Use the coefficients on level j to compute the coefficients on level $j-1$ as follows:

Step 2a: Compute the scaling coefficient

Case 1: **If** $N^j_k = 1$, **then** set

$$d^{j-1}_{k,0} := d^j_{k',0}$$

Case 2: **If** $N^j_k \geq 2$, **then** set

$$d^{j-1}_{k,0} := \frac{\sqrt{N^j_{k'}}\, d^j_{k',0} + \sqrt{N^j_{k'+1}}\, d^j_{k'+1,0}}{\sqrt{N^{j-1}_k}}$$

Step 2b: **If** $N_k^{j-1} \geq 2$, **then** compute the Haar-like coefficient as

$$d_{k,0}^{j-1} := \frac{\sqrt{N_{k'+1}^{j}}\, d_{k',0}^{j} - \sqrt{N_{k'}^{j}}\, d_{k'+1,0}^{j}}{\sqrt{N_k^{j-1}}}$$

Step 2c: **If** $N_k^{j-1} \geq 3$, **then** compute the Walsh-like coefficients. **For** $\ell = 1, \ldots, 2^{j_{\max}-j} - 1$:

Case 1: **If** neither subregion has a coefficient with tag ℓ, **then** do nothing

Case 2: **If** (without loss of generality) only subregion k' has a coefficient with tag ℓ, **then** set

$$d_{k,2\ell}^{j-1} := d_{k',\ell}^{j}$$

Case 3: **If** both subregions have coefficients with tag ℓ, **then** compute

$$d_{k,2\ell}^{j-1} := \left(d_{k',\ell}^{j} + d_{k'+1,\ell}^{j} \right) \Big/ \sqrt{2}$$
$$d_{k,2\ell+1}^{j-1} := \left(d_{k',\ell}^{j} - d_{k'+1,\ell}^{j} \right) \Big/ \sqrt{2}$$

GHWT on P_6 – Fine-to-Coarse Dictionary

Note that the basis vectors (or coefficients) with tag ℓ on level j were used to generate those with tags 2ℓ and $2\ell+1$ on level $j-1$.

Using this fact, we can reorganize the basis vectors by their tags to yield the *fine-to-coarse dictionary*:

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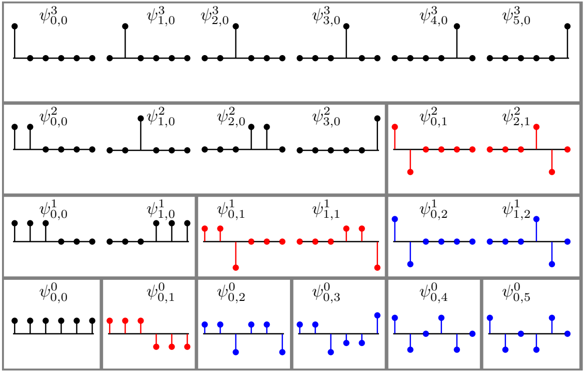
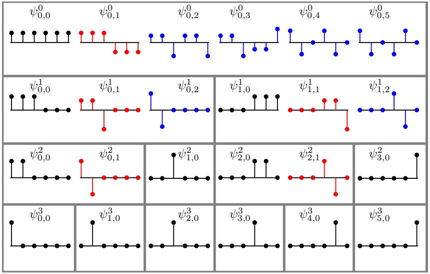
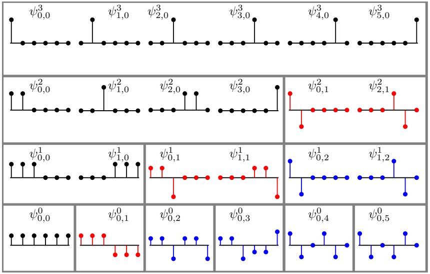


GHWT on P_6



(a) Coarse-to-fine dictionary



(b) Fine-to-coarse dictionary

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- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

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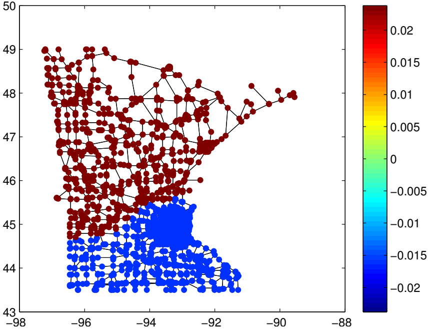
Level $j = 0$, Region $k = 0$, $l = 1$



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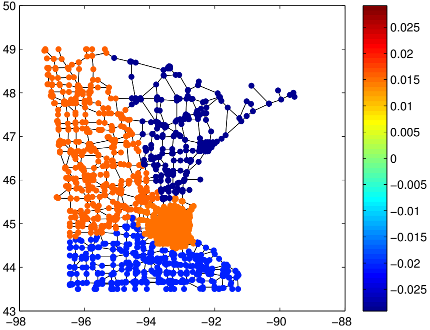
Level $j = 0$, Region $k = 0$, $l = 2$



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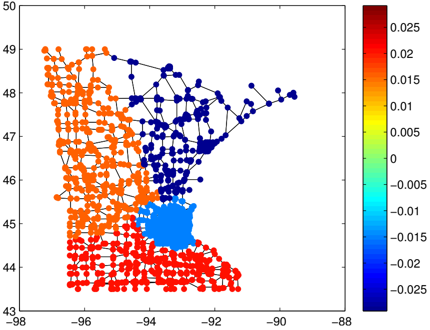
Level $j = 0$, Region $k = 0$, $l = 3$



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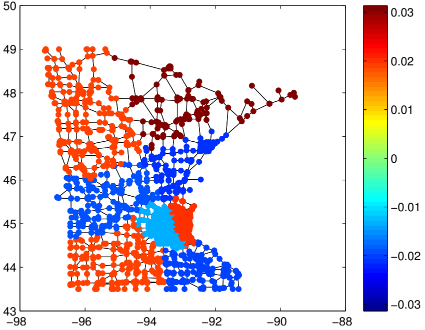

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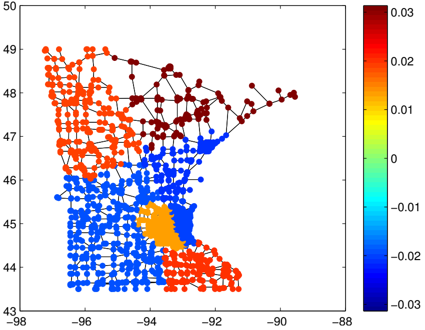
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

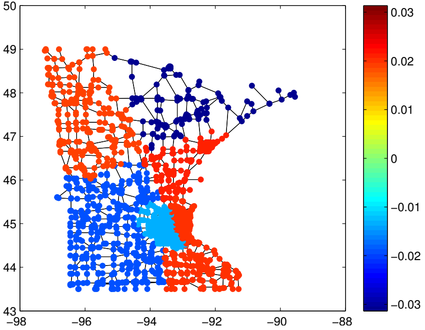
Level $j = 0$, Region $k = 0$, $l = 5$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

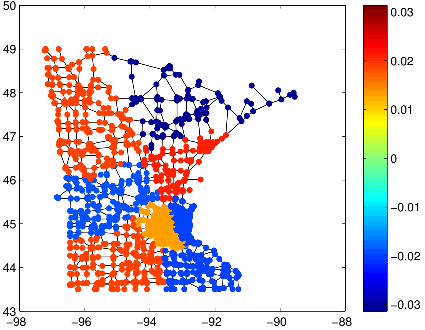
Level $j = 0$, Region $k = 0$, $l = 6$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

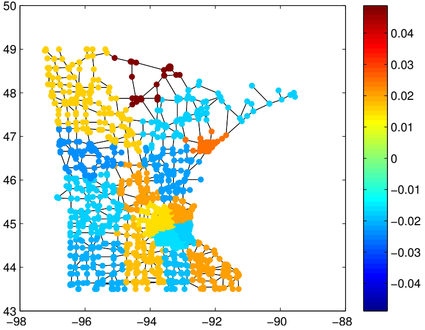
Level $j = 0$, Region $k = 0$, $l = 7$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

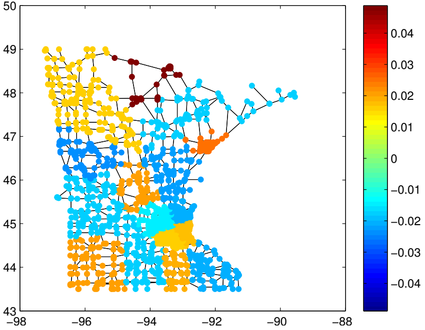
Level $j = 0$, Region $k = 0$, $l = 8$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

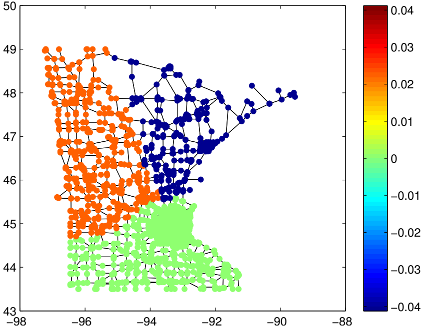
Level $j = 0$, Region $k = 0$, $l = 9$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

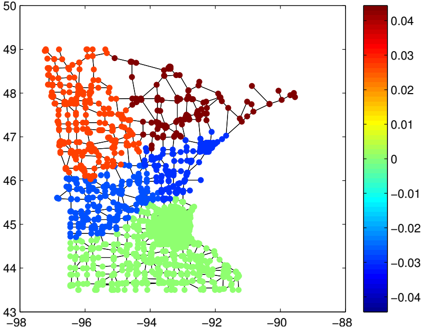
Level $j = 1$, Region $k = 0$, $l = 1$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

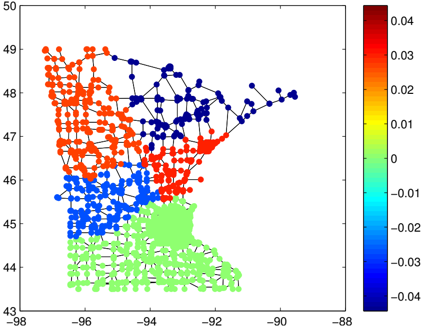
Level $j = 1$, Region $k = 0$, $l = 2$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
- Inverse Euclidean weights, partitioned via the Fiedler vector of L_{rw}

Level $j = 1$, Region $k = 0$, $l = 3$



GHWT on MN Road Network

- $j = 0$ is the coarsest level, $j = 16$ is the finest
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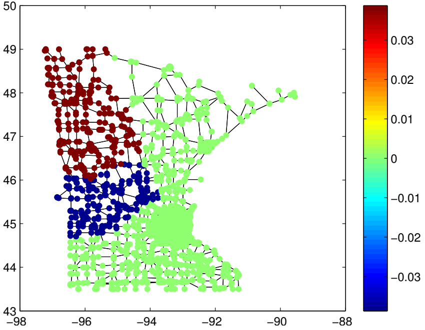
Level $j = 2$, Region $k = 0$, $l = 1$



GHWT on MN Road Network

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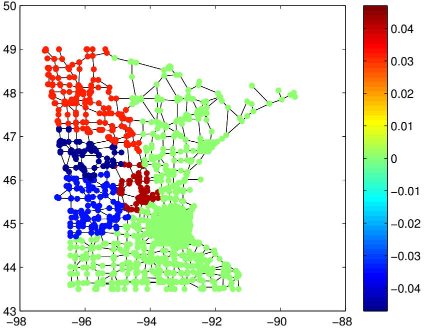
Level $j = 2$, Region $k = 0$, $l = 2$



GHWT on MN Road Network

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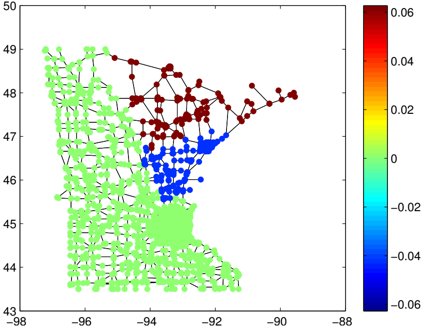
Level $j = 2$, Region $k = 1$, $l = 1$



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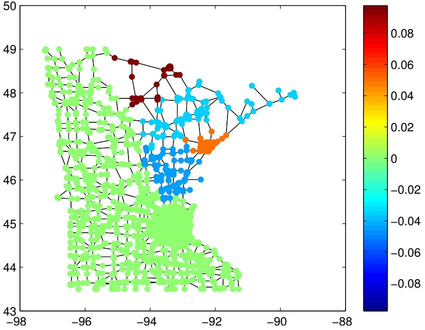
Level $j = 2$, Region $k = 1$, $l = 2$



GHWT on MN Road Network

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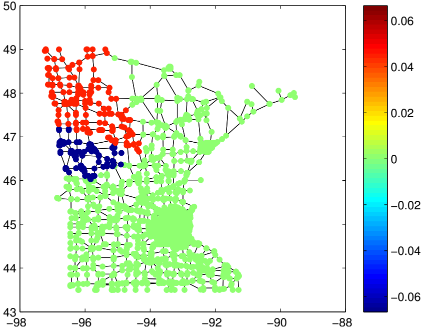
Level $j = 3$, Region $k = 0$, $l = 1$



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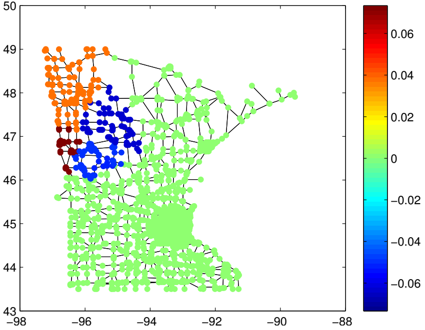

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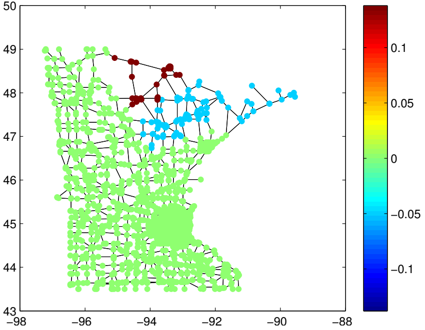

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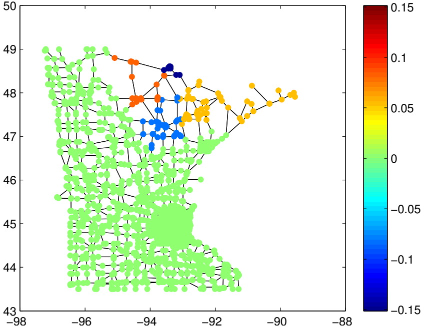
Level $j = 3$, Region $k = 2$, $l = 2$



Observations

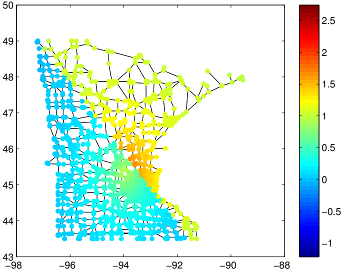
- When performed on an unweighted dyadic path graph (partitioned dyadically), the GHWT corresponds exactly to the Haar-Walsh wavelet packet transform
- The generalized Haar basis is a choosable basis from the fine-to-coarse dictionary
- Given a recursive partitioning with $O(\log N)$ levels, the computational cost of the GHWT is $\boxed{O(N \log N)}$

	N	$j_{\max}$	GHWT Run Time
MN Road Network	$2,636$	14	0.11 s
Facebook Dataset	$4,039$	26	0.62 s
Brain Mesh Dataset	$127,083$	20	4.29 s

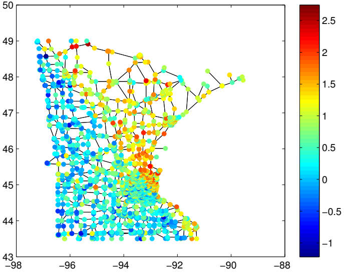
(Experiments performed using MATLAB on a personal laptop.)

Denoising Experiment

1. Added noise to the mutilated Gaussian on the MN road network (left) to yield a signal with an SNR of 5.00 dB (right)



(a) Original signal



(b) Noisy signal

2. Expanded into 3 bases: Laplacian eigenvectors, the generalized Haar basis, and the GHWT coarse-to-fine level $j = 6$ basis
3. Denoised via soft-thresholding, using an elbow detection algorithm to determine the thresholds

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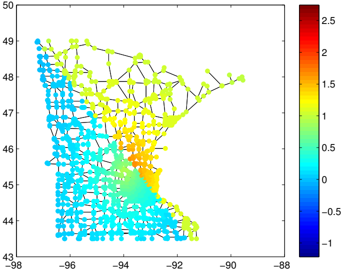

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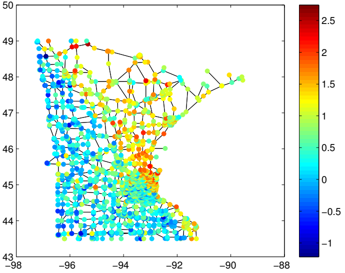

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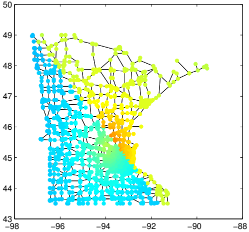

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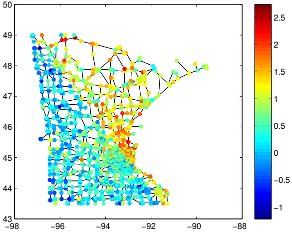
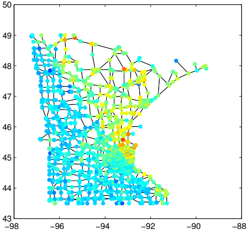
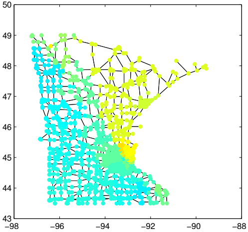


(b) Noisy signal

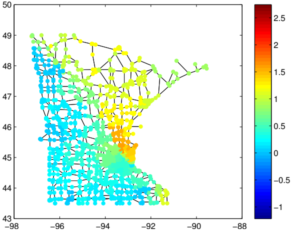
Signal	SNR
Noisy	5.00 dB
Lap. Eigs.	8.29 dB
Haar	10.98 dB
GHWT $j = 6$	13.38 dB



(c) Laplacian eigenvectors



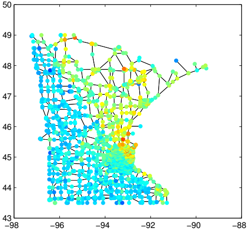
(d) Generalized Haar basis



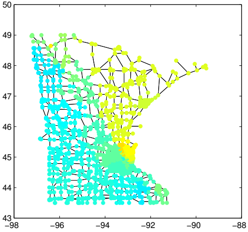
(e) GHWT Coarse-to-Fine Level $j = 6$

Denoising Experiment

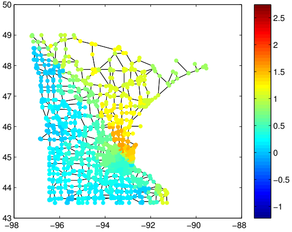
- GHWT level 6 has the best localization for this denoising task
  - Small enough to capture details, large enough to drown out noise
  - The generalized Haar basis is too localized and the Laplacian eigenvectors are too global
- Most retained coefficients for GHWT level 6 are **scaling coefficients**, not **Haar-like** (as in the generalized Haar basis)



(a) Laplacian eigenvectors



(b) Generalized Haar basis



(c) GHWT Coarse-to-Fine Level $j = 6$

Summary

- The GHWT is a *multiscale transform* that is a generalization of the Haar Transform and the Walsh-Hadamard Transform
- It produces an overcomplete dictionary of orthonormal bases from which we can choose a basis most suitable for the task at hand

Future Work

- Allow for partitions to have "soft" boundaries
- Investigate different graph partitioning methods
- Investigate usefulness for classification and function estimation

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Acknowledgements & References

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- NSF VIGRE DMS-0636297
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References

1. J. Irion, N. Saito: "The Generalized Haar-Walsh Transform," *Proceedings of 2014 IEEE Workshop on Statistical Signal Processing*, pp. 488–491, 2014.
2. J. Irion, N. Saito: "Hierarchical Graph Laplacian Eigen Transforms," *Japan SIAM Letters*, vol. 6, pp. 21–24, 2014.
3. R. Coifman, M. Wickerhauser: "Entropy-based algorithms for best basis selection," *IEEE Transactions on Information Theory*, vol. 38, no. 2, pp. 713–718, 1992.